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Structural and stochastic analysis of a zero-dimensional climate system

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SUMMARY

An 'almost trivial' climate system of geometrical dimension zero is analysed, the complexity of which has been reduced to a minimum. It can be simply described as the globally averaged energy flux balance between infrared emission and solar heat input, expanded by a linear albedo–temperature feedback. This nonlinear and time-dependent climate model is formulated as a gradient system of a potential and can be analysed without explicit time integration. It includes many of the results which are also exhibited by one-dimensional energy balance models. Two equilibrium solutions appear. The stable one is characterized by the interglacial, whereas the unstable equilibrium defines a lower bound for temperature (state variable) changes which the system can absorb. Beyond a threshold of an external parameter combination (fold catastrophe) no equilibria exist so that the system attains a 'deep freeze' climate situation. A -2 power law describes the linear response of the (internally stable) system to weather fluctuations.

1. INTRODUCTION

There has recently been continuous growth in the number and complexity of models simulating the earth's climate. Most of these approaches towards a climate system analysis can be classified into two categories.

(1) A hierarchy of climate models can be established according to the dimensions of the geometrical space occupied by the system (e.g. Schneider and Dickinson 1974). It has been claimed that little progress will be made in understanding more complex models unless simpler types in the hierarchy are thoroughly analysed. Therefore, many studies concentrate on one-dimensional energy balance models of the climate system (Budyko 1969; Sellers 1969; Faegre 1972; Gordon and Davies 1974; North 1975; Frederiksen 1976; Chyleh and Coakley 1975; Ghil 1976; etc.).

(2) The other classification is based on the methodology of analysis by which the time-related behaviour of the climate model is investigated (Lorenz 1975). Predictions of the first kind describe the behaviour of the internal climate variables at fixed boundary conditions (external parameters). Predictions of the second kind characterize the changes of the climate system due to the influence of external parameters.

The purpose of this paper is two-fold: A climate system of geometrical dimension zero is described (section 2) belonging to the 'geometrical hierarchy'. This climate system is analysed with respect to its structural behaviour (sections 3 and 4) and its stochastic response to short-period internal fluctuations (sections 5 and 6).

2. CLIMATE SYSTEM OF GEOMETRICAL DIMENSION ZERO

The global energy balance is determined by a conservation law which, if integrated over the total mass (per unit area) of the climate system, yields

$$c . dT/dt = R\downarrow - R\uparrow \qquad (2.1)$$

The storage term (left) is balanced by the net radiation (right) which consists of the net incoming solar radiation $R\downarrow$ and the net outgoing longwave emission $R\uparrow$.

The internal (or state) variable, T, of the global climate system is interpreted as the average temperature of an ocean on a spherical planet which is subject to radiative heating. This allows the storage term to be simplified by a constant thermal inertia $c > 0$ described by a mixed ocean layer of fixed depth h and area coverage $\alpha = 70{\cdot}8\%$:

$$c = \rho_w c_w h \alpha \quad . \quad . \quad . \quad . \qquad (2.2)$$

The net incoming solar radiation

$$R\downarrow = \tfrac{1}{4}\mu I_0(1-\alpha_p) \quad . \quad . \quad . \qquad (2.3)$$

is modified by an external model parameter $\mu > 0$ to allow for variability in the solar constant, I_0. The planetary albedo α_p incorporates an albedo–temperature feedback similar to Budyko (1969) and Sellers (1969)

$$\alpha_p = a - bT \quad . \quad . \quad . \quad . \qquad (2.4)$$

The external parameters $a, b > 0$ define a linear feedback between ice and albedo variability and temperature T. In contrast to Sellers's one-dimensional model a nonlinear feedback cutoff (α_p = const. beyond a prescribed temperature threshold) cannot implicitly be incorporated in the globally integrated model unless a two-compartment system is constructed. However, if the linear relation (2·4) leads to complex temperature values, or if the temperature of the systems tends to (minus) infinity, a limiting value for the planetary albedo has to be introduced to cut the feedback off and to keep the system stable. As will be shown in section 3, only one upper bound for the albedo $\alpha_p = 0{\cdot}75$ has to be defined.

The net outgoing longwave radiation $R\uparrow$ is prescribed by a Stefan-Boltzmann law

$$R\uparrow = \varepsilon\sigma T^4 \quad . \quad . \quad . \quad . \qquad (2.5)$$

The effective emissivity $\varepsilon = \varepsilon_e(1-\varepsilon_a)$ can be interpreted such that it includes the surface emissivity ε_e and transmissivity $(1-\varepsilon_a)$ of the overlying atmosphere; counter-radiation from the atmosphere is parameterized in terms of the surface temperature T.

Equations (2.1) to (2.5) lead to a nonlinear differential equation for temperature T of the climate system:

$$\begin{aligned} dT/dt &= f(T; a, b, c, \varepsilon, \mu) \\ &= (1/c)\{-\varepsilon\sigma T^4 + \tfrac{1}{4}\mu I_0 bT + \tfrac{1}{4}\mu I_0(1-a)\} \quad . \quad . \qquad (2.6) \end{aligned}$$

The external parameters $x = (a, b, c, \varepsilon, \mu)$ determine the behaviour of this simple climate system which has geometrical dimension zero and is characterized by only one internal climate state variable, T. In the following sections the behaviour of $f(T)$ is analysed, where it can be helpful to interpret $f(T)$ as a gradient system of the potential $V(T)$ with $dV/dT = -f(T)$.

3. Equilibria, internal and structural stability

The equilibrium (steady state) solutions of Eq. (2.6),

$$f(T_e) = 0 \quad . \quad . \quad . \quad . \qquad (3.1)$$

reduce the climate system to a bi-quadratic polynomial representing a fold line emerging from a particular section of a 'swallow tail' catastrophy (see, e.g., Thom 1975; Woodcock and Poston 1974):

$$-(c/\varepsilon\sigma)(dT/dt) = T_e^4 - pT_e + q = 0 \quad . \quad . \qquad (3.2)$$

where $p = (\tfrac{1}{4}\mu I_0/\varepsilon\sigma)b > 0$, and $q = -(\tfrac{1}{4}\mu I_0/\varepsilon\sigma)(1-a) > 0$.

It appears that the equilibrium solutions are not influenced by the five external parameters $x = (a, b, c, \varepsilon, \mu)$ independent of each other but by two combinations of them (p,q) which do not involve c. Considering only the physically realistic roots of Eq. (3.2) one obtains

$$T_e^{\pm}(x) = A \pm (\tfrac{1}{4}p/A - A^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad (3.3)$$

where $A = (\frac{1}{3}q)^{\frac{1}{2}}\cosh\{\frac{1}{3}\ln(B+(B^2-1)^{\frac{1}{2}}\}$ and $B^2 = (\frac{1}{3}q)^{-3}(\frac{1}{4}p)^4$.

The equilibrium solutions (3.3) describe a surface in (T_e, p, q) space consisting of two branches T_e^+ and T_e^- coalescing at the bifurcation line $\overset{*}{T}_e$ (Fig. 1). The bifurcation is sufficiently determined for $\frac{1}{4}p/A - A^2 = 0$ (or $\overset{*}{T}_e = A$), which is fulfilled only if $B^2 = 1$. These two conditions lead to the following system of equations describing a hypersurface in (T_e, p, q) space, which is discussed in some detail later:

$$\begin{aligned} 0 &= -T_e^3 + \tfrac{1}{4}p \\ 1 &= (\tfrac{1}{3}q)^{-3}(\tfrac{1}{4}p)^4 \end{aligned} \quad . \quad . \quad . \quad (3.4)$$

An example of the equilibrium solutions is presented for the following external parameter combination which will also serve as the reference situation (subscript o) in the following sections:

$$x_o = [a_o = 2{\cdot}8;\ b_o = 0{\cdot}009\,\mathrm{K}^{-1};\ \varepsilon_o = 0{\cdot}69;\ \mu_o = 1]$$

or equivalently $[p_o = 78{\cdot}2 \times 10^6\mathrm{K}^3;\ q_o = 156{\cdot}4 \times 10^8\mathrm{K}^4]$. These values are chosen to yield the globally averaged 'present day' (reference) temperature: $T_{eo}^+ = 288{\cdot}6\,\mathrm{K}$. They are also in agreement with other one-dimensional energy balance models (e.g. Ghil 1976) and zonally averaged observations (Cess 1976).

The steady-state solutions (3.3) depend on the external parameter values. Most commonly discussed are temperature variations $T_e(\mu)$ due to changes in the relative intensity, μ, of solar radiation. With all other parameters fixed and prescribed by 'present day' conditions, one obtains the results shown in Fig. 1. Simultaneously, the results can be interpreted to depend on changes of the emissivity ε^{-1} if $\mu = 1$, due to the external parameter combination in p and q. The two equilibrium solutions $T_e^+(\mu)$, $T_e^-(\mu)$ represent two climatic states where

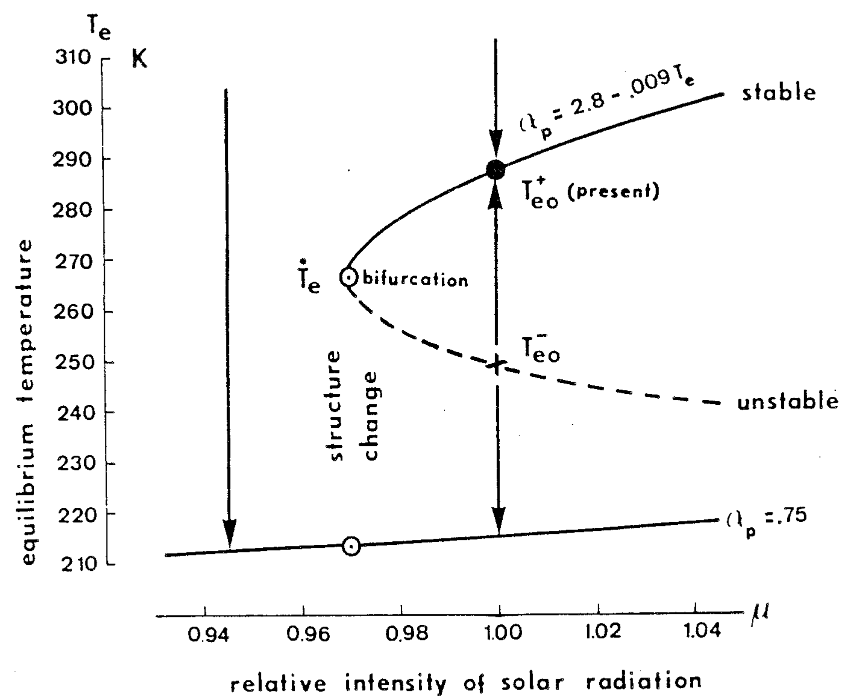

Figure 1. Equilibrium solution $T_e(\mu)$ of the zero-dimensional climate system depending only on changes in relative intensity of solar radiation.

the upper branch may be identified as the interglacial – the lower branch will be interpreted later. The bifurcation $\overset{*}{T}_e(\mu) = 266{\cdot}67\,\text{K}$ at $\mu = 0{\cdot}97$ is attained for a solar constant I_0 diminished by 3%. Beyond this value the 'deep freeze' climate of the limiting planetary albedo $\alpha_p = 0{\cdot}75$ appears as the only (and trivial) equilibrium solution of Eq. (2.6) for $b = 0$, $a = \alpha_p$. However, far-reaching conclusions drawn from this almost trivial or qualitative model must be interpreted with care, especially as $\alpha_p = 0{\cdot}75$ has been chosen arbitrarily.

The internal stability of the equilibrium solution T_e characterizes the time-dependent behaviour of the system due to variations of the internal variable T for fixed external parameters. A criterion of internal stability (instability) can be deduced from the linearized version of the basic climate equation (2.6) which is obtained by a truncated Taylor series expanded about the equilibrium $f(T_e) = 0$:

$$\begin{aligned} dT/dt &= f(T_e) + df/dT|_{T_e}(T - T_e) \\ &= -\lambda(T - T_e) \qquad . \quad . \quad . \quad . \end{aligned} \tag{3.5}$$

Internal stability (instability) is defined by the negative (positive) real part of the eigenvalue, $-\lambda$ (see also Eq. (3.4)):

$$-\lambda = df/dT|_{T_e} = (4\varepsilon\sigma/c)(-T_e^3 + \tfrac{1}{4}p) \begin{matrix} < 0 \text{ stable} \\ > 0 \text{ unstable} \end{matrix} \qquad . \tag{3.6}$$

Fig. 2 presents a graph of Eq. (3.6) showing the internal stability condition projected on the (T_e, p) plane for a representative parameter interval. The 'present day' condition (indicated by a heavy dot) appears as the internally stable solution determined by the reference parameters x_o.

The inverse of the eigenvalue $\tau_c = \lambda^{-1}$ defines the (e-folding) time scale according to

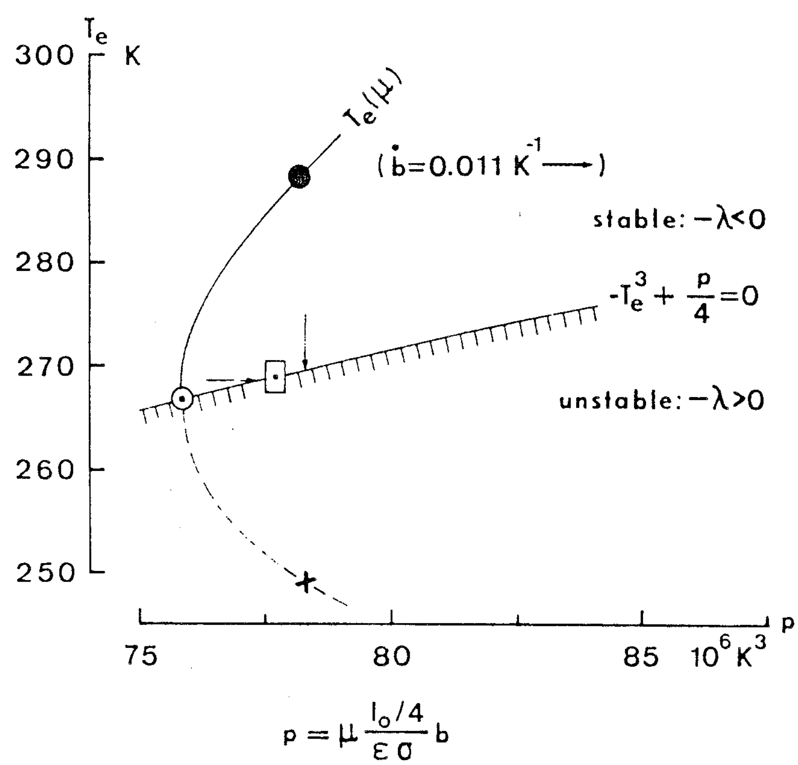

Figure 2. Internal stability diagram. The symbols refer to the structural analysis: variation of a: ↓ ; and b: →; minimum distance ⊡; bifurcation ⊙; linear instability $\overset{*}{T}_e = T^{\pm}_{eo}(\rightarrow)$; reference condition: stable ●, unstable ×. The equilibrium solution $T_e(\mu)$ has been added.

which the linearized climate system approaches (leaves) the stable (unstable) equilibrium, i.e. where the nonlinear system behaves locally like a linear one. For an ocean of mixed layer depth 50 m (Eq. (2.2)) one obtains a typical present-day ($x = x_o$) time scale $\tau_{co} \sim 7$ yr. Comparing conditions (3.6) with (3.3) it follows that the upper branch T_e^+ represents the only stable equilibrium, i.e. the sink which the time-dependent solutions of an initial-value problem approach; whereas the unstable equilibrium solution T_e^- will never appear as a solution of time-dependent numerical calculations (Sellers 1969; Gal-Chen and Schneider 1976; etc.).

Discussing the nonlinear stability of the time-dependent climate system (2.6) it appears that the unstable equilibrium solution T_e^- (lower branch in Fig. 1) represents a lower bound for initial values from which temperature trajectories start leading towards the stable equilibrium T_e^+. For initial values $T < T_e^-$ the time-dependent temperature flow tends towards minus infinity unless the albedo–temperature feedback (Eq. (2.4)) is cut off and replaced by a constant albedo ($\alpha_p = 0{\cdot}75$; Fig. 1). This ensures a lower bound of the temperature ('deep freeze'). For a fixed external parameter constellation the 'stable basin' of the climate system is represented by $T > T_e^-$, where all initial values have trajectories towards the stable equilibrium T_e^+. This can directly be visualized because the climate equation (2.6) appears as a gradient system (i.e. the gradient of a potential; see, e.g., Hirsch and Smale 1974), the equilibrium ($f = 0$) of which defines the necessary condition for the extrema of the potential; and the sign of the first derivative (i.e. the eigenvalue $-\lambda$) sufficiently defines these extrema as a local minimum (stable equilibrium) neighbouring a maximum (unstable equilibrium). Beyond both extrema the potential tends towards (plus and minus) infinity. Thus, for fixed parameters there occurs no temperature oscillation of the climate system but a temperature flow which orthogonally crosses the level surface of the potential. This flow is directed towards the only stable equilibrium (interglacial), or towards the 'deep freeze' situation if the initial values ($T < T_e^-$) lie beyond the unstable equilibrium T_e^- (i.e. the local maximum of the potential).

A condition for the existence of any equilibrium (fixed point) at all may be called the problem of *external stability* because it depends on the external parameter combination. If there are no equilibria, the temperature of the system drops to minus infinity for any initial condition. Therefore, the upper bound for the planetary albedo ($\alpha_p = 0{\cdot}75$) has to be introduced to ensure the lower bound for the temperature ('deep freeze'). This guarantees continuity with the initial-value problem for temperatures below the internally unstable equilibrium T_e^- (see Fig. 1). The transition between the parameter region with fixed points (equilibria) to the one without is given by the 'structural instability', more specifically, by a fold catastrophy. This criterion for a bifurcation can be simply derived by a combination of the internal stability condition ((3.6) for $\lambda = 0$) and the equilibrium climate equation (3.2) eliminating the internal variable T_e (see also Eq. (3.4)):

$$B^2 = (\tfrac{1}{3}q)^{-3}(\tfrac{1}{4}p)^4 = (\tfrac{1}{4}\mu I_0/\varepsilon\sigma)(\tfrac{1}{4}b)^4\{-3/(1-a)\}^3 = 1 \quad . \qquad (3.7)$$

For $B^2 > 1$ two equilibria exist, the internal stability of which has been discussed (Eq. (3.6); Fig. 2). For $B^2 < 1$ there is no equilibrium, due to an imaginary root in A, Eq. (3.3). Figure 3 shows the graph of the structural instability (fold) line as projected on to the (p,q) plane; it defines the bifurcation line separating the parameter regions with and without fixed points (equilibria).

Again, this can be interpreted by the potential of the gradient system (2.6) the local maximum and minimum of which coincide at the fold (bifurcation line) and vanish beyond it.

It appears that, once the system lies beyond the threshold values of the external parameter combination ($B^2 < 1$) and the internal variable ($T < T_e^-$), the outgoing radiation $R\uparrow$ (to which the highest power in T is attached) dominates the system. Thus, the longwave

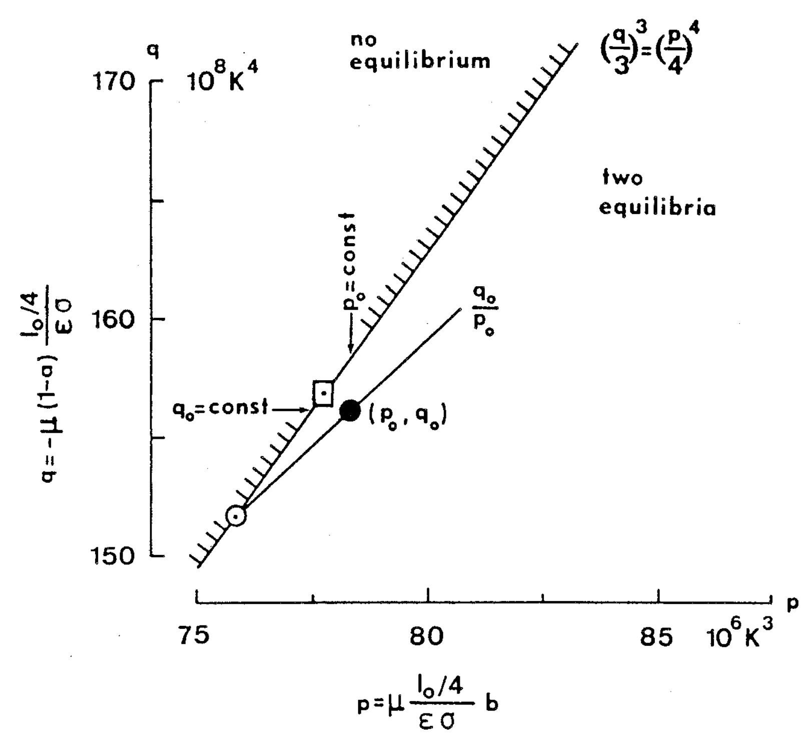

Figure 3. Bifurcation diagram; for symbols see Fig. 2.

emission exceeds the albedo–temperature feedback process, which in turn reduces the net incoming solar radiation $R\downarrow$ by too high an albedo. This leads to a negative temperature tendency (in Eq. (2.6)) which has artificially to be stopped by the 'deep freeze' situation. Within these threshold values there is the stable climate basin with one stable equilibrium solution as a result of the steady-state balance between the radiative energy fluxes; the unstable equilibrium (including the bifurcation) denotes the boundary of the basin.

4. Sensitivity and structural analysis

Some applications of this simple climate model will be discussed mainly under the aspect of 'all other factors being constant . . .' (Bryson 1968), except the one external parameter under examination (see also Flohn 1969). The other external parameters remaining unchanged are assumed to take the values of the 'present day' (reference) situation (subscript o).

On the above premises the sensitivity of the stable equilibrium climate system is conveniently defined as a measure of the reaction of the internal climate variable T_e to relative changes of one of the external parameters x:

$$\beta_x = dT_e^+/d(\ln x)|_{x_o} \quad . \quad . \quad . \quad (4.1)$$

Such a sensitivity parameter β_x can be simply derived from Eq. (3.2) for any $x = a,b,c,\varepsilon,\mu$. It can be shown that its magnitude increases with decreasing distance of the (stable) equilibrium $T_e^+(x)$ (Eq. (3.3)) from the related bifurcation $\overset{*}{T}_e(x)$ (Eqs. (3.4) or (3.6) (3.7)).

The distance between a bifurcation and an equilibrium solution is indicated by a trajectory of the equilibrium solution in the (T_e,p,q) space caused by continuous changes in the external parameters. Projections of these trajectories can be traced in the (T_e,p) or the (p,q) plane (Figs. 2 and 3).

The following five examples illustrate the applicability of this simple model, with respect to a structural analysis of the 'present day' situation of the climate system:

(i) q_o = const. Variability in p with q fixed describes changes which depend on the albedo–temperature feedback parameter b only. The bifurcation (and structural change) due to the p (or b) variation is determined by Eqs. (3.6), (3.7) or (3.4) as indicated in Figs. 2 and 3 by a horizontal arrow: $\overset{*}{T}_e(\overset{*}{b}) = 268{\cdot}71\,\mathrm{K}$; $\overset{*}{b} = 0{\cdot}0089\,\mathrm{K}^{-1}$ or $\overset{*}{p} = 77{\cdot}6\times10^6\mathrm{K}^3$; respectively. This can be visualized by the intersection of the plane q_o = const. in (T_e,p,q) space with the hypersurface (3.4); the related projections are shown in Figs. 2 and 3. The sensitivity of the present-day climate with respect to changes in b yields $\beta_b = \mu I_0 b T_e/4c\lambda = +1280\,\mathrm{K}$, i.e. a 1% change in b would produce a 12·8 K change in T_e unless the bifurcation is reached. (Note that $c\lambda$, as defined by Eq. (3.6), is independent of the thermal inertia c.)

(ii) p_o = const. Variability in q with p fixed describes changes which are caused by the other albedo–temperature feedback parameter a. The related bifurcation and the sensitivity parameter are $\overset{*}{T}_e(\overset{*}{a}) = 269{\cdot}39\,\mathrm{K}$, $\overset{*}{a} = 2{\cdot}81$, (or $\overset{*}{q} = 158\times10^8\mathrm{K}^4$), $\beta_a = -\mu a I_0/4c\lambda = -1380\,\mathrm{K}$ (Figs. 2, 3; vertical arrow). The large sensitivities β_a, β_b clearly indicate the proximity of the reference situation to structural instability.

(iii) q_o/p_o = const. This ratio defines the gradient of a straight line in the (p,q) plane connecting the origin ($p = 0$, $q = 0$) with the 'present day' parameter combination (p_o,q_o). This gradient is identical to the ratio $-(1-a_o)/b_o = 200\,\mathrm{K}$; the related straight line is defined by μ (or ε^{-1}) variations with a_o, b_o unchanged. In (T_e,p,q) space the intersection of the plane p_o/q_o = constant with the hypersurface yields $\overset{*}{T}_e = 266{\cdot}67\,\mathrm{K}$, $\overset{*}{\mu} = 0{\cdot}97$ (for $\varepsilon_o = 0{\cdot}69$), or $\overset{*}{\varepsilon} = 0{\cdot}71$ (for $\mu = \mu_o = 1$). These are the same results as presented in section 3 for an equilibrium temperature depending on the solar radiation $T_e(\mu)$, but they are seen from a different point of view. The sensitivity of the 'present day' climate with respect to μ (or ε) variations is $\beta_\mu = -\beta_\varepsilon = +\varepsilon\sigma T_e^4/c\lambda = 393\,\mathrm{K}$.

(iv) $T_{eo}^+ = \overset{*}{T}_e$. In this case the external climate parameters are changed such that the surface temperature T_{eo}^+ becomes structurally unstable (i.e. a bifurcation point). Introducing $T_{eo}^+ = \overset{*}{T}_e = 288{\cdot}6\,\mathrm{K}$ into Eqs. (3.4) or (3.6), (3.7) one obtains the bifurcation parameters $\overset{*}{p} = 95{\cdot}55\times10^6\mathrm{K}^3$, $\overset{*}{q} = 206{\cdot}39\times10^8\mathrm{K}$ which lie outside the parameter interval selected for the instability diagrams (Figs. 2, 3). The albedo–temperature feedback parameters have to be changed to $\overset{*}{a} = 3{\cdot}38$, $\overset{*}{b} = 0{\cdot}011\,\mathrm{K}^{-1}$ assuming emissivity $\varepsilon = \varepsilon_o$ and solar radiation $\mu = \mu_o$, fixed. This is equivalent to an increase of b_o or p_o by 22% and of $(1-a_o)$ or q_o by 32%, i.e. quite large variations on a global scale. It is obvious that the 'present day' equilibrium temperature T_{eo}^+ cannot become structurally unstable unless both a and b are simultaneously changed.

Investigating the Budyko type of climate model with linearized longwave emission $R\uparrow$, however, the stability analysis is confined to the condition (3.5) of linear (internal) stability at fixed $T_e = T_{eo}^+$. In this case the system becomes unstable ($-\lambda < 0$) if $b > 0{\cdot}011\,\mathrm{K}^{-1}$; but a remains unconsidered. It is this instability value which is discussed by Lemke (1977) and Held and Suarez (1974) using Budyko's one-dimensional energy balance model, and which can well be compared with the results from the zero-dimensional model. The above considerations find direct application in sections 5 and 6, where the linearized model version is investigated.

(v) Minimum parameter variation. The final goal is to find the external parameter combination $(\overset{*}{p},\overset{*}{q})_{\min}$ of a bifurcation which is closest to the 'present day' climate, where all parameters are allowed to change. The condition to be fulfilled is

$$\{(\overset{*}{p}_{\min}-p_o)^2+(\overset{*}{q}_{\min}-q_o)^2\}^{\frac{1}{2}} = \text{minimum} \quad . \quad . \quad (4.2)$$

In combination with the equations describing the hypersurface (3.4) and applying variational principles, one obtains a fifth-order polynomial:

$$\overset{*}{T}_e^5 - \tfrac{4}{3}\overset{*}{T}_e^3 - \tfrac{1}{3}q_o\overset{*}{T}_e - \tfrac{1}{3}p_o = 0 \quad . \quad . \quad . \qquad (4.3)$$

The roots of Eq. (4.3) can be calculated by an appropriate numerical algorithm. There appear only two real solutions which determine the bifurcation at minimum and maximum (p,q) distance from the given reference situation (p_o,q_o). The requested parameter combination of minimum distance yields $\overset{*}{T}_e = 268{\cdot}67\,\mathrm{K}$ with $\overset{*}{p}_{\min} = 77{\cdot}56 \times 10^6\mathrm{K}^3$, $\overset{*}{q}_{\min} = 156{\cdot}29 \times 10^8\mathrm{K}^4$ (see Figs. 2, 3; open square). This can be achieved by simultaneously changing a, b such that $(\overset{*}{q}/\overset{*}{p})_{\min} = 201{\cdot}5\,\mathrm{K}$; i.e. only slight external changes are needed to bring the present situation toward the structural instability.

These five examples demonstrate how a structural analysis can be applied to the zero-dimensional climate system. As stated before, this model is merely qualitative as are all other simple climate models which also show the well-known sensitivity to small changes in external parameters. Additionally, the zero-dimensional system reflects all the most important results of one-dimensional climate models (Sellers 1969; Gal-Chen and Schneider 1976; Ghil 1976, etc.), namely their structural behaviour; the mathematical background to achieve these results is elementary and the agreement is not only qualitative.

5. Linearized climate system with stochastic forcing

In this section the analysis of the zero-dimensional climate system is continued without an explicit time integration of the model. Instead, the linear version (3.5) of the climate model (2.6) is investigated as a system which has a long-term response to a short-period and stochastic forcing due to weather fluctuations. These processes are not included in the (macroscale) parameterizations of the slowly changing climate system (section 2). For a more refined description the action of these (microscale) weather fluctuations has to be added to the climate system in terms of a stochastic forcing w (Hasselmann 1976) which includes all the short-period dynamic, energetic, and radiative mechanisms involved. With such a stochastic expansion Eq. (3.5) can be formulated

$$dy/dt = -\lambda y + w \quad . \quad . \quad . \qquad (5.1)$$

where $y = T - T_e$. The negative eigenvalue $(-\lambda < 0)$ appears as the stabilizing feedback parameter which characterizes the response time scale $\lambda^{-1} = \tau_c$ of the linearized climate system.

Given the stable reference situation T_{eo}^+, the linearized system (5.1) becomes unstable $(-\lambda > 0)$ for $b > 0{\cdot}011\,\mathrm{K}^{-1}$ as derived by substituting $T_{eo}^+ = 288{\cdot}6\,\mathrm{K}$ into Eq. (3.6) (see also section 4). Eq. (5.1) resembles the well-known Langevin equation which is an example of the so-called stochastic (or random) equations describing, e.g., the Brownian motion problems as a one-component system. The Langevin type of climate equation (5.1) can be solved analytically if the following assumptions are incorporated (see, e.g., Balescu 1975):

(i) The weather fluctuations w are assumed to be irregular, i.e. separated in time and statistically independent. Consequently, the auto-covariance with respect to an ensemble average, $\langle\,\rangle$, of weather fluctuations at two times, $P(\tau) = \langle w(t)w(t+\tau)\rangle$, exists only for time intervals of the order of the duration of the fluctuations $\tau_w \ll \tau_c$.

(ii) A (spectral) gap in the variance (density) is assumed to separate the time scales of weather and climate fluctuations so that the effect of the irregular weather fluctuations can be introduced into the responding climate system as white noise: $D = \tfrac{1}{2}\int_{-\infty}^{+\infty} P(\tau)\,d\tau$.

This two-timing concept allows the following stochastic description by avoiding the deterministic detour; this leads to the variance of the climate system responding to irregular weather fluctuations:

$$R(\tau) = (D/\lambda)\{1-\exp(-2\lambda|\tau|)\}, -\lambda < 0 \quad . \quad . \quad (5.2)$$

which gives a smoothed picture of the stochastically forced system (5.1) with the details of microscale processes averaged out. The stochastic forcing realized by the climate system becomes effective after time scales $\tau > \tau_c$ when the initial values are forgotten as time progresses.

It is convenient to deduce the Fourier transform $G(\omega)$ of (5.2) which can be compared with power spectra, evaluated from observed climate temperature variances. With the δ-function expression $\delta(\omega) = \lim_{\lambda \to 0} (\lambda/\pi)(\lambda^2+\omega^2)^{-1}$, the Fourier transform $G(\omega)$ of the variance $R(\tau)$ yields

$$G(\omega) = (D/\lambda)\{\delta(\omega)-(2/\pi)(4\lambda^2+\omega^2)^{-1}\}$$
$$= (D/\pi)(\lambda^2+\omega^2)^{-1} \text{ for large } \tau. \quad . \quad . \quad (5.3)$$

The second expression is valid for large response times $\tau \gg +\lambda^{-1}$ because $\exp(-2\lambda|\tau|) \ll 1$ can be neglected in Eq. (5.2). Thus, the temperature variance spectrum of the climate model increases with decreasing frequency according to a -2 power law, but it flattens at lower frequency ($\omega < \lambda$) because of the negative feedback. This result has also been derived by Hasselmann (1976) based on similar but more complicated considerations. At zero frequency the maximum spectral variance density mainly depends on the stabilizing feedback parameter $-\lambda$ and, of course, on the white noise level D, to be specified in the following section.

6. Stochastic analysis and application

A few relationships can be derived to give additional insight into the stochastically forced climate system with linear feedback:

(i) No feedback ($\lambda = 0$). In the case that the negative feedback vanishes (e.g. $b \to 0{\cdot}011\,\mathrm{K}^{-1}$) the variance of the internal climate variable grows linearly with increasing response time, due to the continuous stochastic forcing, and becomes infinite. Taylor series

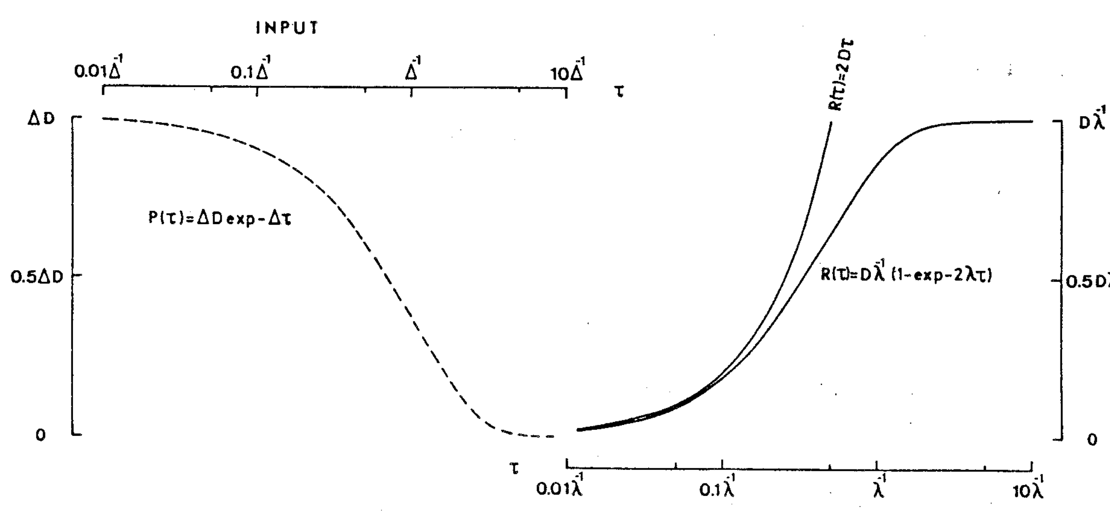

Figure 4. Normalized input auto-covariance and response variance of stochastically forced climate system; magnitudes of an application are given in section 6.

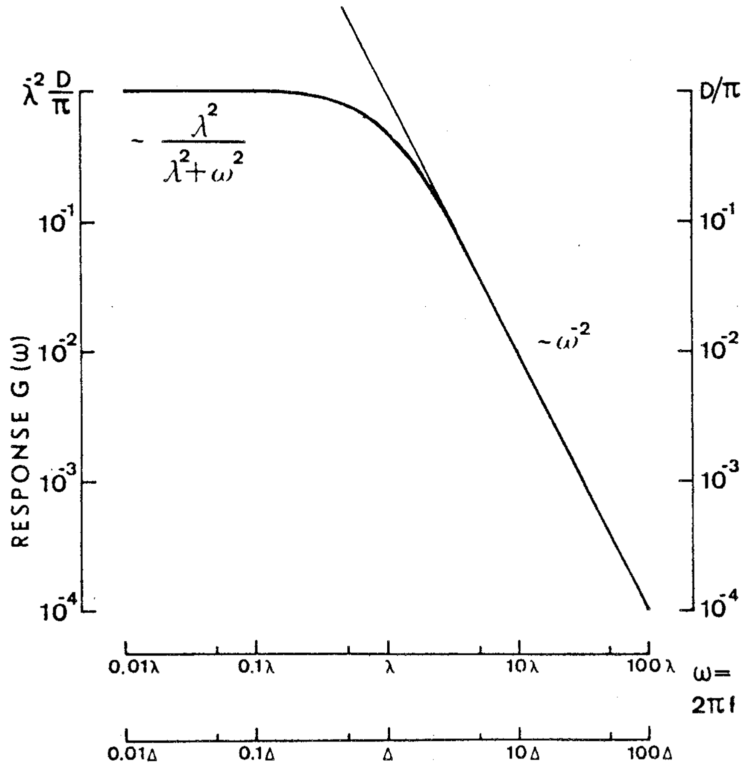

Figure 5. Normalized input and response spectra of stochastically forced climate system; magnitudes of an application are given in section 6.

expansion of Eq. (5.2) leads to the related variance and its spectral density distribution which are independent of λ:

$$R(\tau) \sim 2D\tau, \quad G(\omega) \sim (D/\pi)\omega^{-2} \qquad (6.1)$$

The same relationships hold for $\tau \to 0$ (or $\omega \to \infty$) (see Eqs. (5.2) and (5.3)); i.e. initially the responses are similar for the climate models with and without feedback. However, after the time scale $\tau_c = \lambda^{-1}$ has been exceeded, the long-term climate feedback starts to influence the shape of the variance as indicated in Fig. 4, which also holds for the frequency $\omega = \lambda$ if the spectral variance density distribution is considered (Fig. 5).

(ii) Maximum variance and spectral density. The upper bounds of the variance of the linear feedback model are simply obtained with the condition $\tau \to \infty$, equivalently the spectral level at zero frequency ($\omega \to 0$):

$$R(\tau = \infty) = D/\lambda, \quad G(\omega = 0) = (D/\pi)\lambda^{-2} \qquad (6.2)$$

These are the scaling factors used in Figs. 4 and 5.

(iii) Stochastic input. For a given climate state the global weather fluctuations are assumed to be represented by a linear, first-order Markov process:

$$dw/dt = -\Delta w \qquad (6.3)$$

Again, the stabilizing feedback parameter $-\Delta < 0$ is a measure of the characteristic time scale of the short-term fluctuations $\tau_w = \Delta^{-1} \ll \lambda^{-1}$. With the solution $w = w_o\exp(-\Delta t)$ of Eq. (6.3) the auto-covariance $P(\tau)$ of the Markov process and its Fourier transform, $F(\omega)$, i.e. the red noise spectrum, can be deduced:

$$\begin{aligned} P(\tau) &= \Delta D\exp(-\Delta|\tau|) \\ F(\omega) &= (D/\pi)\Delta^2(\omega^2+\Delta^2)^{-1}. \end{aligned} \qquad (6.4)$$

The Fourier transform (6.4) of the Markov type of weather fluctuations has the same shape as the climate response, also representing a red variance spectrum. The most important distinction between the spectra can be made by the time scales involved, which differ significantly: $\tau_c = \lambda^{-1} \gg \tau_w = \Delta^{-1}$. Therefore, the weather fluctuations behave as a white noise input spectrum on which the Langevin-type climate model responds. The coefficient D characterizing this white noise level can now be specified by the Markov process (red noise) weather fluctuations:

$$D = \tfrac{1}{2}\int_{-\infty}^{+\infty} P(\tau)\, d\tau = P(\tau = 0)/\Delta = \langle w'^2 \rangle \tau_w \quad . \tag{6.5}$$

where '∞' is small (large) compared with the climate time scale $\tau_c = \lambda^{-1}$ (weather time scale $\tau_w = \Delta^{-1}$). Thus, $\tau_w = \Delta^{-1}$ also appears as an integral correlation time scale of the weather process, the total variance of which, $\langle w'^2 \rangle$, is determined by the ensemble average of the weather fluctuations at a given climatic state.

(iv) Application. The magnitudes of the stochastic input parameters (6.5) are needed to determine the response of the climate system. They are based on the following scaling arguments. The total energetic variance of the stochastic weather process $c^2\langle w'^2 \rangle = \sigma_w^2$ depends on the r.m.s. deviation of a globally averaged radiative energy flux divergence which is assumed to be comparable to the flux divergence itself (see also Lemke 1977). The latter is defined by that part of the incoming energy which is efficiently used by the climate heat engine: $\eta R\downarrow$ (efficiency $\eta = 0{\cdot}01$; e.g. Lettau 1954); the dissipative processes occur at the short-period end of the variance spectrum. With $\sigma_w = \eta R\downarrow = 3\,\mathrm{W\,m^{-2}}$ one obtains the variance $\langle w'^2 \rangle = 10^{-2}\mathrm{K^2 yr^{-2}}$ where the thermal inertia of the system, Eq. (2.2), is given by $c = 1{\cdot}5 \times 10^8\,\mathrm{kg\,K^{-1}\,s^{-2}}$, representative for a well-mixed water reservoir of depth 50 m covering 70·8% of the earth's surface.

The integral correlation time scale of the weather fluctuations will be prescribed by the period of vacillation (or blocking activity) because of its large, occasionally even hemispheric, extent as it appears more appropriate for the globally averaged conditions considered; thus $\tau_w = \Delta^{-1} = 25\,\mathrm{d}$. Another possible choice is the transient eddy lifetime of about a week.

The time scale and the variance of the weather fluctuations prescribe the white noise level $D = 2{\cdot}7 \times 10^{-2}\mathrm{K^2 yr^{-1}}$ as the stochastic forcing of the climate system. The response to this input also depends on the climate time scale $\tau_c = \lambda^{-1} = 7\,\mathrm{yr}$ specified by the 'present-day' situation x_o and the thermal inertia $c = 1{\cdot}5 \times 10^8\,\mathrm{kg\,K^{-1}\,s^{-2}}$, Eq. (3.6). Now the scaling parameters (6.2) of Figs. 4 and 5 can be derived: the response auto-covariance of the temperature $\lambda^{-1}D = 0{\cdot}75\,\mathrm{K^2}$, the related maximum spectral variance density $\lambda^{-2}D/\pi = 1{\cdot}6\,\mathrm{K^2 yr}$. Perhaps a change by an order of magnitude for the stochastic input D may not be too unrealistic. These values given, the stochastic weather fluctuations lead to the total r.m.s. surface temperature deviation $(2D\lambda^{-1})^{\frac{1}{2}} = 0{\cdot}61\,\mathrm{K}$ of the globally averaged climate system from its stable equilibrium T_{eo}^{+}. (For an ocean layer depth 25 m, $(2D\lambda^{-1})^{\frac{1}{2}} = 0{\cdot}86\,\mathrm{K}$.) This value may be compared with 1·1 K for a stochastically forced one-dimensional energy balance model (Lemke 1977). It should be mentioned that the variances of zonally averaged temperatures are higher than those of globally averaged temperatures. Both variances can be exceeded by the variance of an individual temperature record, for which Kutzbach and Bryson (1974) obtain r.m.s. deviations ranging from 0·4 to 1·2 K. The same argument holds for the spectral variance densities. The calculated global value is small compared with the solar radiation sensitivity $\beta_\mu = 393\,\mathrm{K}$ (section 4, for $q_o/p_o = \mathrm{const.}$).

Finally, it should be added that the slope of the spectrum compares well with the variance spectra of temperature on time scales of 10 to 1000 yr, as derived by Kutzbach and Bryson (1974) from instrumental, historical, and botanical records.

7. Conclusion

An almost trivial climate system has been introduced as a gradient system, and its qualitative behaviour is studied using basic and well-known methods:

The structural analysis characterizes this climate system as one of the seven 'elementary catastrophes' according to which complex natural phenomena with sudden changes can be described mathematically (Thom 1975).

The stochastic analysis treats this climate system in terms of a Langevin equation analogous to a classical problem of non-equilibrium statistical mechanics.

Some of the most important features obtained from these analyses are:

(i) Two equilibrium solutions appear: the internally stable one belongs to a stable climate basin where the system is resilient; i.e. it can absorb changes. Outside the basin the system shows catastrophic behaviour where it approaches the (somewhat artificially introduced) 'deep freeze' situation. The boundary of the resilient basin is defined with respect to: (a) changes of the internal (state) variable (global temperature) which should not drop below the second (unstable) equilibrium solution; and (b) changes of the external parameters, which should not be situated on and beyond the bifurcation line, at which the equilibria coincide. The physical interpretation of this behaviour is based on longwave emission, which dominates incoming radiation, if the temperature–albedo feedback becomes too strong.

(ii) The response of the linearized climate model exhibits a red -2 power law of the spectral temperature variance density, which is due to continuous stochastic forcing by short-period weather fluctuations.

(iii) The qualitative and quantitative results can well be compared (and compete) with investigations of more complex one-dimensional energy balance models, although much information (e.g. the meridional energy distribution) disappears due to the global averaging process. It may be concluded that such information is only of marginal importance for the structural behaviour of simple energy balance models. The simplicity of the zero-dimensional climate system has the advantage that its treatment and the results obtained are transparent.

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Appendix

List of symbols

T; t	temperature (internal variable); time
$R\downarrow$, $R\uparrow$	radiative fluxes
μI_0; α_p	solar radiation, I_0 solar constant; planetary albedo
σ	Stefan-Boltzmann constant
a, b	albedo–temperature feedback coefficients
ε	emissivity
c; c_w, ρ_w, h, α	thermal inertia; specific heat, density, depth and area cover of an ocean layer
x	external parameters: a, b, c, ε, μ
p, q, A, B	combinations of external parameters
f; V	climate equation: gradient system; potential
λ	eigenvalue (feedback parameter)
$\tau_c = \lambda^{-1}$	climatic time scale
$\tau_w = \Delta^{-1}$	weather time scale
w	weather fluctuations
$y = T - T_e$	new internal variable
β_x	sensitivity parameter
τ; ω	correlation time; frequency
R; P	(covariance) variance of response; Markov-process (auto-covariance)
G; F	spectral variance density of response; input
$\langle\,\rangle$; D	weather ensemble average; white noise level
δ	delta-function
η	thermodynamic efficiency

Suffixes and indices

e; o	equilibrium; reference (present day) climate
c; w	climate; weather processes
min	minimum parameter variation
*	bifurcation