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# On the distribution of cloud top heights based on stochastic forcing

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ABSTRACT

Irregular small-scale motions, which are introduced into a one-dimensional entraining jet lead to a random cloud with additive stochastic forcing. These clouds are characterized by probability distributions of the thermodynamic cloud properties. Applying a first-passage analysis, probability distributions for maximum cloud top heights can be deduced. In the disturbed tropical atmosphere, random clouds reveal a bimodal probability distribution of cloud top heights, which is essentially due to the mid-tropospheric minimum of the moist static energy. They represent deep and shallow convection. If separated, each scale may be approximated by a log-normal distribution. Some aspects of the competition between these two cloud scales are evaluated.

## 1. Introduction

In the tropical atmosphere, diagnostic studies and data analyses of large-scale variables suggest a bimodal distribution of clouds in disturbed weather systems (e.g., Ogura and Cho, 1973; Yanai et al., 1973); i.e. there are deep and shallow cumuli, but there is a gap for clouds terminating in the mid-troposphere (McBride, 1981).

Theoretical investigations on this problem seem to be confined to the parameterization of convection; Ooyama (1971) represents a cloud ensemble by an unspecified dispatcher (or cloud spectrum distribution) function which Arakawa and Schubert (1974) couple with the large-scale motion by the cloud work function. This approach connects the cloud statistics with large-scale variables, whereas Fraedrich (1977) prescribes the structure (but not the parameters) of cloud distributions as independent of the large scale. A few studies treat the non-linear interaction between individuals of a cloud ensemble: non-precipitation clouds are simulated by Beniston and Sommeria (1981) showing reasonably good agreement with parameterization schemes; van Delden and Oerlemans (1982) investigate cloud grouping with a hydrodynamic model. In contrast to these studies, a stochastic approach is described by Cho (1978), who leaves the random clouds independent of one another, to yield a Poisson distribution.

In the following, a different approach of stochastic cloud modelling is suggested to obtain a simple explanation for the bimodal cloud distribution in the tropics without prescribing two distinct cloud radii. The entrainment equation, which describes the deterministic part of the cloud process, is extended by stochastic forcing to parameterize smaller scale motions (Section 2) which had been removed in formulating the purely deterministic entrainment equation. Probability distributions of thermodynamic cloud properties and of cloud top heights are evaluated for an ensemble of random clouds (Section 3). An example (Section 4) describes a bimodal cloud top height distribution as it occurs in the equatorial trough zone. Finally, a sensitivity analysis (Section 5) gives some information on the competition of deep and shallow clouds.

## 2. A one-dimensional cloud and its stochastic perturbation

The one-dimensional cloud is described by a cylindrical entraining jet of constant cross-sectional area. The basic equations are the mass and energy balance

$$\frac{\partial \rho w}{\partial z} + \nabla \cdot \rho v = 0, \tag{2.1}$$

$$w \frac{\partial h}{\partial z} + v \cdot \nabla h = 0, \tag{2.2}$$

where the moist static energy $h = c_p T + gz + Lq$ is a conserved quantity (see list of symbols). Combination yields

$$\frac{\partial \rho w h}{\partial z} + \nabla \cdot \rho v h = 0. \tag{2.3}$$

Averaging over a vertically constant cloud area "$A$" one obtains for the mass balance (2.1)

$$\frac{\partial \overline{\rho w}}{\partial z} + \overline{\nabla \cdot \rho v} = 0, \tag{2.4a}$$

where $\overline{\rho w} = A^{-1} \int_A \rho w \, \mathrm{d}f$ denotes the area-averaged vertical cloud mass flux, which is often approximated by a top hat profile. The area averaged horizontal divergence (2.4) may be transformed by the flux $\rho c$ normal to the boundary $L$ of the cloud (Gaussian theorem)

$$\overline{\nabla \cdot \rho v} = \frac{1}{A} \int_A \nabla \cdot \rho v \, \mathrm{d}f = \frac{1}{A} \oint_L c\rho \, \mathrm{d}l = \frac{L}{A} \widetilde{c\rho}, \tag{2.4b}$$

where "$\sim$" denotes the boundary average.

Area averaging of the moist static energy balance (2.3) leads to a similar equation, which may analogously be transfromed by the Gaussian theorem:

$$\frac{\mathrm{d}\overline{\rho w h}}{\mathrm{d}z} + \frac{L}{A} \widetilde{c\rho h} = 0. \tag{2.5}$$

Both the vertical and lateral cloud flux $\overline{\rho w h}$ and $\widetilde{c\rho h}$ can be decomposed into a mean and an eddy term:

$$\overline{\rho w h} = \overline{\rho w}\bar{h} + \overline{(\rho w)' h'},$$
$$\widetilde{c\rho h} = \widetilde{c\rho}\tilde{h} + \widetilde{(c\rho)^* h^*}. \tag{2.6}$$

Average and deviation are related to cloud area $A$ ("—" and " ' ") and lateral boundary $L$ ("$\sim$" and "*").

The mean fluxes define the deterministic part of the cloud process. The cloud area averaged vertical fluxes of mass, $\overline{\rho w}$, and moist static energy ($\bar{h} = h_c$): $\overline{\rho w}\bar{h} = \overline{\rho w} h_c$ are balanced by the lateral inflow of mass, $c\rho$, which is associated with entrainment of environmental moist static energy ($\tilde{h} = h_e$): $\widetilde{c\rho}\tilde{h} = \widetilde{c\rho} h_e$. The turbulent fluxes $\overline{(\rho w)' h'}$ and $\widetilde{(c\rho)^* h^*}$ are formally defined by covariances; they appear as additive eddy fluxes in terms of averages over cloud area and boundary, which vary with height $z$, etc.

The mean and eddy fluxes (2.6) can be introduced into the energy balance (2.5) and combined with the mass balance (2.4). This leads to a cloud process, where both deterministic and turbulent contributions are explicitly incorporated:

$$\frac{\mathrm{d}h_c}{\mathrm{d}z} = \frac{1}{\overline{\rho w}} \left\{ \frac{\mathrm{d}\overline{\rho w}}{\mathrm{d}z} (h_e - h_c) - \frac{\mathrm{d}}{\mathrm{d}z} \overline{(\rho w)' h'} - \frac{L}{A} \widetilde{(\rho c)^* h^*} \right\}, \tag{2.7}$$

i.e., the cloud process is described by the vertical variation of the area-averaged moist static energy of the cloud, $\bar{h} = h_c$, which depends on the deterministic and turbulent flux divergences, i.e., on cloud scale and sub-cloud scale processes (see Augstein et al., 1980).

### 2.1. *The deterministic cloud process (entraining jet)*

Neglecting the turbulent fluxes in (2.7) leads to the well-known entrainment equation, which characterizes the deterministic aspects of the cloud; i.e. there is only entrainment ($\bar{h} = h_c$) up to cloud top ($\hat{z}$) and no overshoot:

$$\frac{\mathrm{d}h_c}{\mathrm{d}z} = \lambda(h_e - h_c). \tag{2.8}$$

This equation describes how environmental air $h_e$ modifies the cloud process $h_c(z)$ by entrainment

$$\lambda = \frac{\mathrm{d} \ln \overline{\rho w}}{\mathrm{d}z}.$$

For constant environmental conditions, $h_e =$ const., an integral or e-folding scale height of clouds can be evaluated, which is inversely proportional to the entrainment factor ($\lambda^{-1}$). The cloud process is deterministic in the sense that lateral convergence entrains environmental air, which modifies the rising cloud by mixing. Additionally, the cloud mass flux increases exponentially, if there is no lateral detrainment.

In the following, it is assumed that the cloud growth comes to an end at the top $\hat{z}$, where cloud and environmental moist static energy are equal:

$$h_c(\hat{z}) - h_e(\hat{z}) = 0.$$

To obtain different top heights of deterministic clouds, one has to assume a spectrum of entrainment rates, which are traditionally parameterized by the cloud radius: $\lambda = b/R$ with $0.16 \leqslant b \leqslant 0.22$.

However, this essentially deterministic approach does not describe the complete cloud process. Turbulent motions, which occur in a cloud-average sense, are not explicitly treated in the derivation of the entrainment formula (2.8). Fluctuating with height, these small-scale motions modify the vertical distribution of the cloud property $h_c$ and consequently the cloud depth $\hat{z}$. These fluctuations can have various sources and consequences: in-cloud turbulence changes the lateral convergence, the entrainment $\widehat{(\rho c)^* h^*}$ and the vertical fluxes $\overline{(\rho w)' h'}$. Additionally, turbulence in the environment may lead to variations of the profile $h_e(z)$, which can be enhanced by an entrainment of previously detrained cloud air into the same or neighbouring cloud.

### 2.2 *A stochastic cloud process (random cloud)*

If turbulent fluctuations and their contributions to the cloud fluxes $\overline{(\rho w)' h'}$ and $\widehat{(\rho c)^* h^*}$ are random, they can be explicitly incorporated into the entrainment equation (2.8) by an additive stochastic forcing term $\xi(z)$. This defines a random cloud which adds an extra inhomogeneous part to the differential equation (2.8):

$$\frac{\mathrm{d}h_c}{\mathrm{d}z} = \lambda(h_e - h_c) + \xi. \qquad (2.9)$$

Now, the cloud property $h_c(z)$ is no longer a deterministic but a random variable of a stochastic cloud process. Formally, the stochastic forcing $\xi$ may be interpreted as a parameterization of the turbulent flux divergences, which are normalized by the mean vertical cloud mass flux (2.7):

$$\xi: -\frac{1}{\overline{\rho w}} \left\{ \frac{\mathrm{d}}{\mathrm{d}z} \overline{(\rho w)' h'} + \frac{L}{A} \widehat{(c\rho)^* h^*} \right\}.$$

Here it is obvious that the stochastic term should be introduced into (2.8) as an additive and not a multiplicative forcing.

In the following, the stochastic forcing $\xi(z)$ is assumed to be white noise, i.e., Gaussian distributed with zero mean and vertically delta-correlated with independent values at each point

$$\langle \xi(z) \rangle = 0.$$

$$\langle \xi(z) \cdot \xi(z + z') \rangle = B\delta(z'), \qquad (2.10)$$

where the brackets $\langle\ \rangle$ denote ensemble averaging and $\delta(z')$ is the delta function.

Although these assumptions are rather restrictive, they are expected to describe the situation satisfactorily as they do in Brownian motion and stochastic climate models (Hasselmann, 1976; Nicolis and Nicolis, 1981). Due to their local character, these random fluctuations rapidly lose their memory of the state which prevailed when they occurred, i.e., they are independent of each other. To obtain different cloud top heights ($h_e = h_c$), it is no longer necessary to prescribe a spectrum of entrainment rates, but to use a representative $\lambda$-value instead, which characterizes the preferred mode of large-scale convective instability. Now, it is the explicit introduction of random forces, which leads to a distribution of maximum cloud top heights (Section 3).

For practical purposes, the intensity $B$ of the stochastic (white noise) fluctuations must be evaluated. If the $\delta$-function is replaced by a rapidly decreasing exponential autocovariance

$$\langle \xi(z)\, \xi(z + z') \rangle = \sigma_{\xi\xi}^2 \exp - |z'|/R, \qquad (2.10a)$$

one obtains the e-folding reduction at the correlation height or length $R$, which defines the vertical scale of stochastic fluctuations. To guarantee their stochastic nature in relation to the cloud process, the length scale $R$ should be significantly smaller than the vertical depth or scale height ($\lambda^{-1}$) of the clouds, i.e. $R \ll \lambda^{-1} \sim 5R$. This is the case if one assumes the cloud radius (which is about five times smaller than the cloud depth) as a characteristic correlation length.

Relating the variance $\sigma_{\xi\xi}^2$ of the $\xi$-fluctuations to the variance $\sigma^2$ of moist static energy $h$, gives

$\sigma^2_{\xi\xi} = \sigma^2/R$. Integration of eq. (2.10) and (2.10a) provides a measure of the intensity $B$, which characterizes the variance of white noise $h$-fluctuations per height:

$$B \sim \frac{2\sigma^2}{R}. \tag{2.10b}$$

A first estimate of the standard deviation $\sigma$ is the moist static energy difference between cloud and environment ($\sim$5–10 J g$^{-1}$); the radius of thermals ranges from 0.1 to 10 km. Although this $\sigma$-estimate may vary with height, the forcing variance $B$ will be treated as a constant parameter in the following discussions.

Without explicitly solving the complete cloud process (2.9), probability densities $p = p(h_c, h_{c0}, z, z_0)$ can be determined. They describe transitions from the state $h_{c0}$ at $z_0$ (backward variables) to a state $h_c$ at a greater height $z$ (forward variables). The Fokker-Planck or Kolmogoroff forward equation (e.g. Arnold, 1973; van Kampen, 1981; Gardiner, 1983)

$$\frac{\partial p}{\partial z} + \frac{\partial}{\partial h_c}\{\lambda(h_e - h_c)\cdot p\} - \frac{\partial^2}{\partial h_c^2}\left(\frac{B}{2}p\right) = 0 \tag{2.11}$$

determines how the transition probability density $p$ of the cloud moist static energy evolves with height depending on drift and diffusion (second and third terms). The backward variables $(h_{c0}, z_0)$ serve as boundary conditions. Here it should be noted that the transition probabilities $p$ are identical with common probabilities, because, due to white noise forcing, we are dealing with a Markov process in a continuous state space, i.e. with continuous realizations. Assuming constant or Gaussian distributed boundary values and white noise fluctuations, the scalar linear stochastic differential equation of type (2.9) represents a Gauss–Markov process; the related probability density $p(h_c, h_{c0}, z, z_0)$ for the cloud static energy $h_c$ is defined by the density of a normal distribution (see, e.g. Arnold, 1973):

$$p(h_c, h_{c0}, z, z_0) = \frac{1}{\sqrt{2\pi K}} \exp -\frac{(h_c - m)^2}{2K}. \tag{2.12}$$

The mean or expectation $m$ is defined by the ensemble average of the solution of eq. (2.9); considering (2.10) yields

$$m = \langle h_c \rangle = h_{c0} \exp\{-\lambda(z - z_0)\} + \int_{z_0}^{z} \lambda h_e(z') \times \exp\{\lambda(z' - z)\}\,\mathrm{d}z', \tag{2.12a}$$

which is identical with the solution of the stochastically unperturbed entrainment equation (2.8). The variance is defined by the ensemble average of the squared deviations from the mean, which are due to the stochastic fluctuations. After some algebra (see e.g. Balescu, 1975) one obtains

$$K = \langle (h_c - m)^2 \rangle = \frac{B}{2\lambda}(1 - \exp\{-2\lambda(z - z_0)\}). \tag{2.12b}$$

The solution (2.12) satisfies the Fokker-Planck equation (2.11).

## 3. The height distribution of clouds—a first passage problem

The first passage problem is defined by the probability distribution of a height $\hat{z}$, where the random variable $h_c(z)$ of the process attains a prescribed value $\hat{h}_c$ for the first time after starting from $h_0$. In this sense, the cloud depth or top height $\hat{z}$ is a first passage height of the cloud process $h_c(z)$ to its top state $\hat{h}_c = h_c(z = \hat{z}) = h_e(z = \hat{z})$ from the cloud base state $h_{c0} = h_c(z = 0)$. A formal set defining the first passage height $z$ of a random cloud $h(z)$ yields

$$\begin{aligned} z = 0: &\quad h_c(z = 0) = h_{c0}, \\ 0 < z < \hat{z}: &\quad h_e(z) < h_c(z) < \infty, \\ z = \hat{z}: &\quad h_c(z = \hat{z}) = \hat{h}_c = h_e(z = \hat{z}). \end{aligned} \tag{3.1}$$

The cloud top state $h_c(\hat{z}) = \hat{h}_c$ can be interpreted as one of the absorbing barriers of the cloud process $h_c(z)$, beyond which cloud growth is not allowed to continue (Section 2). The other absorbing barrier is $h_c = \infty$. The probability that the cloud process remains in $h_e < h_c < \infty$ after rising to height $z$ is

$$\mathrm{prob}\,(\hat{z} \geqslant z) = P(h_{c0}, \hat{h}_c, z) = \int_{\hat{h}_c}^{\infty} p(h_{c0}, h_c, z)\,\mathrm{d}h_c. \tag{3.2}$$

Thus, the probability density of the first passage height, i.e. absorption has not yet occurred (it occurs at $z = \hat{z}$), follows directly:

$$g(z | h_{c0}, \hat{h}_c) = -\frac{\partial}{\partial z} P(h_{c0}, \hat{h}_c, z). \tag{3.3}$$

Thus, $g(z | h_{c0}, h_c)$ is a probability density (with respect to height) of the random cloud process $h_c(z)$

occupying all states $h_c(\hat{z}) \leqslant h_c < \infty$ smaller than or equal to the cloud top state, $\hat{h}_c = h_e(z)$, after starting from $h_{c0}$.

The first passage height distribution can now be derived from (2.12) using (3.3) and the standard definitions (e.g. Johnson and Kotz, 1970) of the Gaussian error integral $\phi$ (cumulative normal distribution function) and its derivative $\phi'$:

$$g(z \mid h_{c0}, \hat{h}_c) = -\frac{\partial}{\partial z} \int_{\hat{h}_c}^{\infty} p(h_{c0}, h_c, z)\, dh_c$$
$$= \frac{\partial}{\partial z} \phi\left(\frac{\hat{h}_c - m}{\sqrt{K}}\right)$$
$$= \phi'\left(\frac{\hat{h}_c - m}{\sqrt{K}}\right) \cdot \frac{\partial}{\partial z}\left(\frac{\hat{h}_c - m}{\sqrt{K}}\right). \tag{3.4}$$

Straight algebra yields

$$g(z \mid h_{c0}, \hat{h}_c) = \frac{1}{\sqrt{2\pi K}} \exp\left\{-\frac{\hat{h}_c - m}{2K}\right\}$$
$$\times \frac{(\hat{h}_c' - m')\, K - (\hat{h}_c - m)\, K'/2}{K}. \tag{3.5}$$

$m$ and $K$ are mean and covariance defined by (2.12), and $m'$, $K'$ are their $z$-derivates. The moist static energy $\hat{h}_c$ (and $\hat{h}_c'$) at cloud top is prescribed by the environmental profile $h_e(z)$ (see Section 2). For $h_e =$ const., the first passage height (3.5) is identical to the first passage problem of a Langevin equation (Wang and Uhlenbeck, 1945, eq. (82)).

The cloud top height distribution $g(z \mid h_{c0}, \hat{h}_c)$ is a measure of the relative frequencies of cloud depths in a cloud ensemble. It depends on the moist static energy of the environment $h_e(z)$, on the intensity $B$ of the stochastic forcing and on the entrainment rate $\lambda$. In Sections 4 and 5, cloud top height distributions are evaluated for tropical and other atmospheric environments.

## 4. Bimodal distribution of cloud top heights in the tropics

The moist static energy profile $h_e(z)$ in the tropics, particularly in the area of the equatorial trough zone, exhibits a significant minimum in the mid-troposphere (see Riehl, 1979, Fig. 2.23). Above cloud base (here at $z = 0$), this profile may be approximated by a quadratic polynomial

$$h_e(z) = h_{e0} + h_{e1}\, z + h_{e2}\, z^2 + \ldots \tag{4.1}$$

The coefficients are defined by the cloud base energy $h_e(z = 0) = h_{e0}$, by the position of the minimum $z_{min}$, where $dh_e/dz = h_{e1} + 2h_{e2}\, z = 0$ and by the magnitude of the minimum $h_{e\,min} = h_{e0} + h_{e1}\, z_{min} + h_{e2}\, z_{min}$. The mean of the stochastically perturbed cloud, $m$, which is identical with the deterministic profile (2.8), can be deduced from (2.12a) in combination with (4.1):

$$m = h_e(z) + (h_{c0} - h_{e0}) \exp(-\lambda z) - \frac{h_{e1}}{\lambda}$$
$$\times (1 - \exp(-\lambda z)) + \frac{2h_{e2}}{\lambda^2}$$
$$\times (-\lambda z + (1 - \exp(-\lambda z))\,. \tag{4.2}$$

$h_{c0}$ is the cloud moist static energy ($h_c(z = 0) = h_{c0}$) at cloud base. Introducing (4.1), (4.2) and (2.12b) into (3.5) leads to the cloud top or first passage height distribution density $g(z \mid h_{c0}, \hat{h}_c = h_e)$ of a cloud ensemble, which represents realizations of individual random clouds.

This statistic is evaluated for a tropical atmosphere under disturbed conditions using the following data and parameters. The moist static energy profile (4.1) is defined by $h_{e0} = 340$ J g$^{-1}$ at cloud base ($z = 0$) and $h_{e\,min} = 330$ J g$^{-1}$ at $z_{min} = 5$ km. All cloud elements start from cloud base with $h_{c0} = 350$ J g$^{-1}$. The cloud radius $R = 1$ km is specified for disturbed weather, which leads to estimates of the entrainment rate, $\lambda = 0.2$ km$^{-1}$ and of the intensity of the stochastic fluctuations, $B = 100$ J$^2$ g$^{-2}$ km$^{-1}$ (see Section 2). With these conditions, the stochastic cloud model produces a bimodal distribution $g(z \mid h_{c0}, h_c)$ of cloud top heights.

(i) A density maximum or mode occurs in upper levels near the termination of the purely deterministic cloud process, where the moist static energy of the entraining jet (2.8) and of the environment meet (Figs. 1a, b). This peak characterizes a relative frequency maximum for deep clouds. For relatively small entrainment rates, deep convection is left almost unaffected by the mid-tropospheric $h_e$-minimum.

(ii) The first peak above cloud base is situated in lower layers. It defines the other density

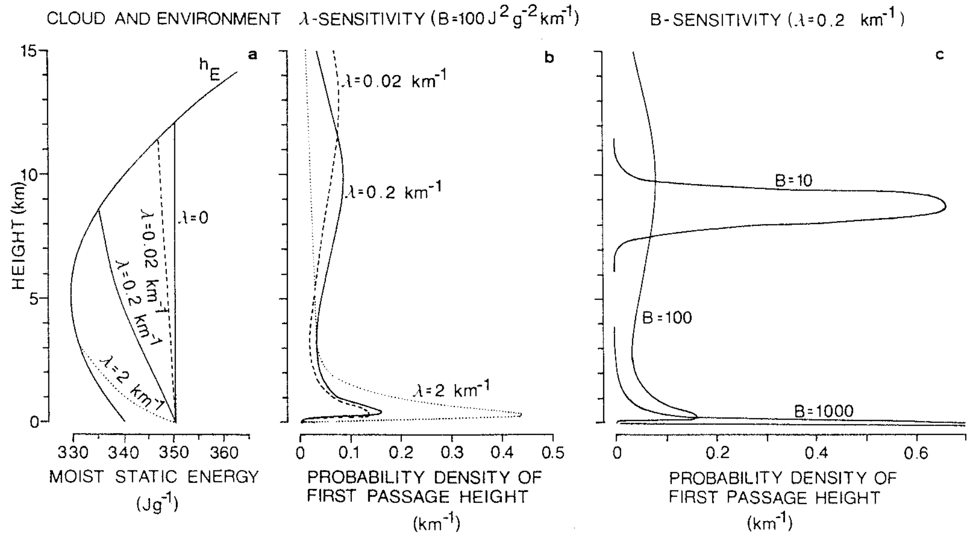


*Fig. 1.* (a) Moist static energy of clouds with varying entrainment rates in a tropical environment $h_e$ (above cloud base). Probability densities of first passage or cloud top heights depending on (b) the entrainment rate ($\lambda$-sensitivity) and (c) on the intensity of the stochastic fluctuations (B-sensitivity).

maximum or mode and represents shallow clouds, which owe their existence to the stochastic forcing.

The strong mid-tropospheric $h_e$-minimum in the tropics separates shallow from deep clouds; i.e., the bimodal feature of the model occurs only in a tropical atmosphere and under disturbed weather conditions, when the cloud ensemble allows deep clouds with small entrainment rates to exist. Both modes unite in lower layers, if the entrainment rates become larger (Fig. 1b) or the stochastic forcing grows (Fig. 1c). Here it should be noted that the shallow random clouds, which are associated with small entrainment rates (i.e. $\lambda \sim 0.02$ km$^{-1}$; Fig. 1b) are mainly due to the turbulent stochastic forcing; this produces the rapid erosion of clouds and thus their shallowness, although they have the (deterministic) tendency to grow to greater depths.

The cloud scale separation in the tropics demonstrates the cooperation of stochastic and deterministic features. Both mechanisms combined describe an individual cloud as a random process (Section 2). The distribution of all realizations of individual random clouds reveals two distinct cloud scales coexisting in the statistics of tropical clouds. It should be noted that it is the application of the first passage height distribution, which unfolds the cloud scale separation from the ensemble of random clouds. The existence of shallow clouds in our model is guaranteed by the stochastic forcing $B$, which, by definition (Section 2), is well separated from the scale of the deterministic cloud process; without stochastic forcing, shallow clouds would not occur. If the entrainment rate $\lambda$, which represents the deterministic impact on the cloud, becomes sufficiently small, deep convection evolves and separates from shallow clouds (Fig. 1). Details follow in Section 5.

For representative tropical clouds ($\lambda = 0.2$ km$^{-1}$, $B = 100$ J$^2$ g$^{-2}$ km$^{-1}$), the cumulative distribution function of cloud top heights (first passage heights) is plotted on log-normal probability paper. Thus, our results can be compared with empirical studies, which claim the log-normality of various cloud properties (e.g. López, 1977), i.e. the normal distribution of the logarithms of cloud top heights. From Fig. 2, it is obvious that the complete distribution cannot be approximated by a log-normal one. But two separate regions may be distinguished approaching log-normality: deep convection (with tops $z \geqslant 7$ or 8 km above cloud base) and shallow convection with tops from above cloud

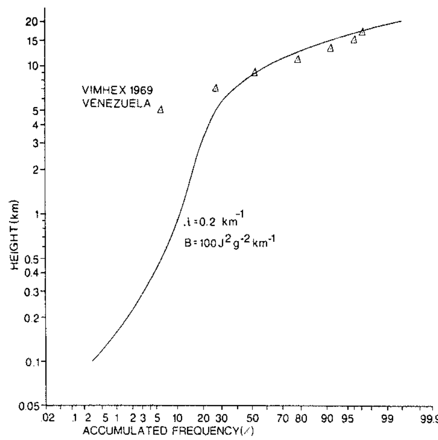


*Fig. 2.* Accumulated frequency distribution of cloud top heights (first passage heights) above cloud base plotted on log-probability paper. Observations are taken from the Venezuelan International Meteorological and Hydrological Experiment (VIMHEX, 1969; Cruz, 1973).

base to the mid-troposphere ($0 \leqslant z \leqslant 7$ or 8 km). Qualitatively, this is in good agreement with observations of maximum top heights of deep clouds over tropical continents (Cruz, 1973). As these observations are based on radar, the deviation in the region of shallow or non-precipitating clouds is not surprising.

## 5. Sensitivity analysis: deep versus shallow clouds

The entrainment factor $\lambda$, the intensity of stochastic forcing $B$ and the profile of the environmental moist static energy $h_e$ affect the cloud top height distribution of a cloud ensemble.

Environmental $h_e$-profiles with simple structure ($h_e =$ const. or linearly increasing with height, i.e. $h_{e1}$ and/or $h_{e2} = 0$ in (4.1) and (4.2)) reveal only a single maximum or mode of the first passage height distribution (Fig. 3). This mode is generated by essentially the same stochastic mechanisms as those by which shallow clouds occur in a tropical atmosphere (Section 4). This peak may be explained qualitatively: with increasing cloud depth, the effect of the stochastic noise accumulates with the height of each individual cloud element. The probability density distribution $p$ of the moist static energy $h_c(z)$ flattens and spreads accordingly. If an individual random cloud is not too far removed from the environmental $h_e$-profile, it attains its cloud top states $\hat{h}_c = h_e$ easily. Thus, a density peak should be expected between $z = 0$ and $z \to \infty$, because clouds start almost unperturbed from cloud base, $g(z = 0) = 0$, and the first passage height distribution diminishes again for increasing depth, $g(z = \infty) = 0$.

The sensitivity of the first passage height distributions (3.5) and their modes is discussed in some detail.

(i) It the stochastic forcing intensity $B$ is reduced (Figs. 1c, 3c), the deterministic part of the cloud process, which is related to the entrainment rate $\lambda$, dominates the ensemble statistic. The modes of the cloud depth distributions lie in the neighbourhood of the purely deterministic cloud top height (Figs. 1a, 3a, 3c for $B = 10$). Vice versa, an increasing stochastic intensity $B$ reduces the deterministically dominated peak. Particularly deep clouds associated with small entrainment rates are reduced and, simultaneously, the frequency or chance of shallow clouds rises (Figs. 1c, 3c for $B = 1000$).

(ii) The interaction or competition between deep and shallow cloud scales can be formulated differently. If the stochastic forcing is kept at a level of sufficient intensity, there are always shallow cloud top heights in the tropical (Fig. 1b) or another atmosphere (Fig. 3b). Leaving the stochastic forcing unchanged, shallow clouds have a greater probability of becoming deep, when the entrainment rate decreases.

In this sense, the chance of deep clouds rises at the expense of shallow clouds. This is well documented in the tropical atmosphere (Fig. 1b), because the bimodal first passage height distribution leads to a clear interpretation of the behaviour of the two distinct cloud scales. For $h_e =$ const., or linearly increasing, the scale competition is less evident (Fig. 3b).

## 6. Conclusion and outlook

Deterministic and stochastic processes are combined in a simple cloud model which leads to probability distributions of the thermodynamic cloud properties and of the maximum cloud top heights. The deterministic part of the cloud de-

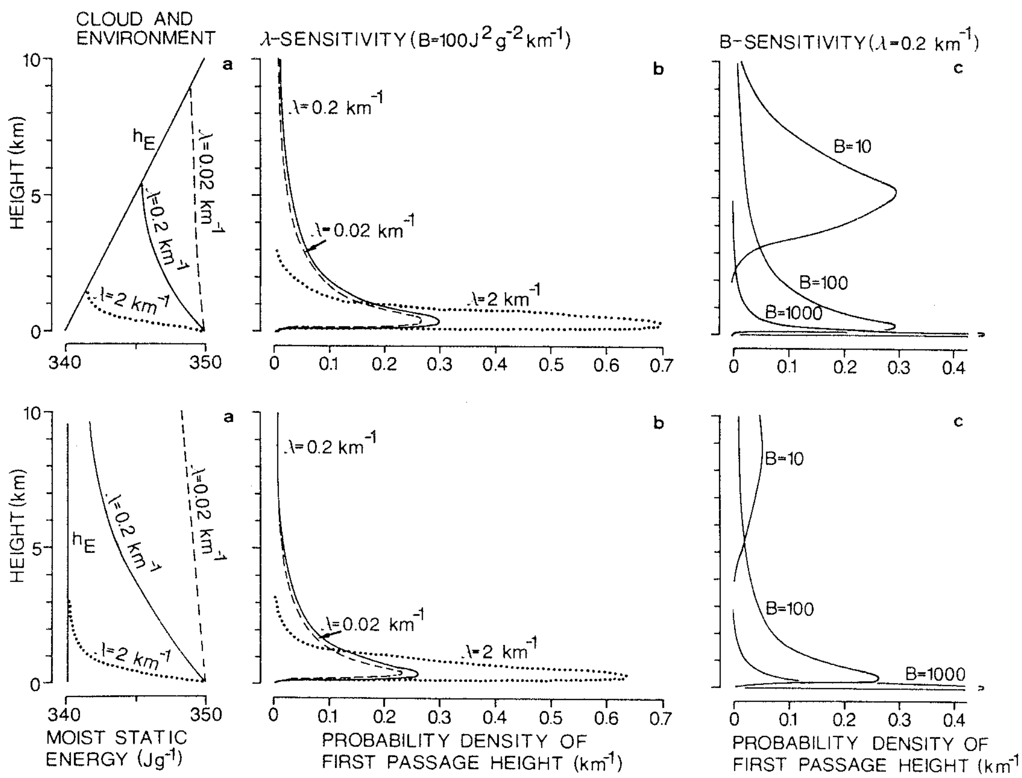


*Fig. 3.* Same as Fig. 1 except for a linearly increasing $h_e$ profile (top) and $h_e$ constant (bottom).

scribes the turbulent entrainment process in an averaged sense, whereas the stochastic part explicitly adds the stochastic fluctuations and their intensity, which have been averaged out from the deterministic model. In this sense, the random clouds produce their own statistic, which depends on the cooperation between these two impacts.

Cloud ensembles in parameterization schemes are commonly represented by deterministic clouds of different size. They are characterized by a distribution of fractional entrainment rates, which are deduced from the large-scale forcing (or closure condition). Their energy, moisture and momentum impact in the large-scale field is defined by source terms, which are produced by compensating subsidence and various forms of detrainment. An ensemble of random clouds is defined by a representative entrainment rate or cloud size and its height distribution is determined by stochastic or turbulent forcing. The characteristic cloud size can be deduced from the preferred mode of large-scale convective instability; the stochastic forcing may be related to a large-scale wind shear. Both of these aspects must be incorporated into a closure condition. The impact of random clouds on the large-scale field remains to be studied; in particular, the appropriate mass flux distribution needs consideration due to its strong influence on the large-scale sources.

As this model simulates statistical features of random clouds, it may offer another approach to the description of cloud ensembles. Of course, comprehensive numerical experiments and observations are needed for a verification of the results.

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## 8. List of symbols

| | |
|---|---|
| $h = c_p T + gz + Lq$ | moist static energy (sensible heat, potential energy, latent heat) |
| $h_e, h_c$ | moist static energy of environment, cloud |
| $z$; $\rho$; $w$, $\boldsymbol{v}$ | height coordinate; density; vertical, horizontal velocity |
| $L, A, R$ | circumference, area, radius of cloud |
| $\lambda, b$ | entrainment rate; empirical factor: $\lambda = b/R$ |
| $\xi, B, \delta(z)$ | stochastic forcing, intensity, delta-function |
| $\sigma_{\xi\xi}^2, \sigma^2$ | variance of $\xi$-fluctuations, of moist static energy |
| $p(h_c, h_{c0}, z, z_0)$; $P$ | (transition) probability density of $h_c$; cumulative distribution function |
| $m, K$ | mean, variance |
| $g$ | density of first passage height distribution |
| $\phi, \phi'$ | Gaussian error integral, derivative |
| suffixes and indices: | |
| —; ′ | average over cloud area $A$; deviation |
| ~; * | average along lateral boundary $L$; deviation |
| ∧; 0 | cloud top; cloud base |
| ⟨ ⟩ | ensemble-average |

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