Estimating Weather and Climate Predictability on Attractors

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ABSTRACT

Predictability is deduced from phase space trajectories of weather and climate variables which evolve on attractors (local surface pressure and a $\delta\,^{18}$O-record). Predictability can be defined by the divergence of initially close pieces of trajectories and estimated by the cumulative distance distributions of expanding pairs of points on the single variable trajectory. The *e*-folding expansion rates characterize predictability time scales. As a first estimate one obtains a predictability time scale of about two weeks for the weather variable and 10–15 thousand years for the climate variable.

1. Introduction

Flows observed in the atmosphere and other hydrodynamic systems reveal a hierarchy of structures ranging from laminar to turbulent motion. Due to the complexity of these flows, they show a sensitive dependence on the initial conditions. Thus, any predictions of weather or climate are accompanied by a growth of errors, which ultimately leads to limits of the predictability. Model flows of various complexity are the most common tools for the study of atmospheric predictability (first introduced by Thompson, 1957). They reveal almost the same quality of prediction (in terms of error doubling times, e.g. Shukla, 1985). Therefore, it may be conjectured that the underlying mathematical structures of the time evolution (which occurs on attractors in phase space) have similar properties. One such property is the dimensionality of attractors, i.e., the number of parameters necessary to control the time evolution in phase space. Some observations suggest a low fractal dimensionality, which—only in a qualitative sense—accounts for the irregular or chaotic behavior of flows. Another property is the rate of divergence of initially close pieces of trajectories evolving on attractors, which will be discussed in this study. It provides a quantitative measure of predictability (i.e., the degree of chaos) and therefore seems to be useful in meteorology. Applying statistical procedures (Grassberger and Procaccia, 1984) we extend the dimensionality analysis (Fraedrich, 1986) to estimates of predictability on attractors using observed weather and climate variables (thereby relaxing the persistence condition, which also included dependent pieces of trajectories). The methodological background is described in section 2, applications are presented in section 3.

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2. Basic concepts

Consider the difference between two states of a deterministic process at a given time. If this difference is small initially and remains small in the future, the deterministic process is stable. Vice versa, if an initially small difference exceeds a threshold value, the process is unstable. Thus, the predictability of dynamic systems is closely related to the problem of stability of the time evolution; a relevant measure of the predictability is the rate at which initially small errors grow. Now consider the time evolution of the weather or climate system. It can be simulated by partial differential equations describing the underlying physical processes. These equations may conveniently be transformed to a set of n ordinary differential equations

$$\frac{d}{dt}x_i = f_i(x_1, \cdots x_n), \quad i = 1, \cdots, n \tag{2.1}$$

with n suitably normalized variables x_i. Thus, the phase space containing the time evolution is spanned by the n different variables x_i, $i = 1, \cdots, n$. A weather state at an initial time is realized by a vector $\mathbf{x}_0 = (x_1(t_0), x_2(t_0), \cdots, x_n(t_0))$ in phase space. Another realization $\mathbf{x}_0 + \delta\mathbf{x}$ may be defined by an initially small vector or deviation from the basic state $\mathbf{x}_0$:

$$\delta\mathbf{x} = (\delta x_1, \delta x_2, \cdots, \delta x_n). \tag{2.2}$$

The difference between the two realizations (or states) can be measured by the distance (Euclidean norm) D:

$$D(t) = (\delta\mathbf{x} \cdot \delta\mathbf{x})^{1/2} \tag{2.3}$$

which evolves as time progresses. Now two routes to predictability experiments can be defined (Lorenz, 1984):

The traditional approach is to solve the nonlinear equations (2.1) twice with slightly different sets of initial conditions. Then $D(t)$ can be evaluated for a sequence of time steps. Beyond the time limit of predictability, $D(t)$ would oscillate about a value not greater than the difference between two randomly selected states of the system. If $D(t)$ stays below this threshold, one can expect predictability for a longer time range.

Another method (Lorenz, 1965) is to evaluate the growth rates of errors δx in the system, which is governed by the set of linear differential equations

$$\frac{d}{dt}\delta x_i = \sum_{j=1}^{n} A_{ij}\delta x_j, \quad i = 1, \cdots, n. \tag{2.4}$$

The coefficients A_{ij} are the elements of the Jacobian matrix of $\mathbf{f} = (f_1, \cdots, f_n)$ defined by the partial derivative of (2.1):

$$A_{ij} = \left.\frac{\partial f_i(x_1, \cdots, x_n)}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0}. \tag{2.5}$$

They are determined at the basic states $\mathbf{x} = \mathbf{x}_0$ changing with each time step; therefore the elements A_{ij} are, in general, time dependent coefficients, which vary with the time evolution $\mathbf{x}(t)$ of (2.1).

If a realization is stable (unstable), $D(\mathbf{x}_0, t)$ remains bounded for all time (or it grows quasi-exponentially). The local stabilities of the weather or climate evolution are determined by the eigenvalues, A_{ij}, (or characteristic exponents) of the Jacobian matrix which change with time. If at least one eigenvalue L has a positive real part, the evolution is unstable and D grows proportionally to $\exp[L(t - t_0)]$; otherwise the solution is stable. The magnitude of the positive characteristic exponents can be used (after appropriate time averaging) as a measure of unpredictability. They define—in a time averaged sense—the mean rate of divergence of initially (i.e., at each time step) close trajectories separated by an infinitesimally small vector $\delta\mathbf{x}$. In this sense they describe the system's sensitive dependence on initial conditions. Thus, the intrinsic unpredictability of the atmosphere is characterized by the magnitude of the positive characteristic exponents to be deduced from observations. Failures of forecast models, however, depend on the sensitivity of their characteristic exponents due to model errors, which will not be discussed here.

The following subsections introduce the concept of characteristic exponents (§2a) and how unpredictability of the weather and climate system can be estimated by the growth rate of infinitesimally small errors (§2b) using observed time series.

a. The characteristic exponent as a measure of predictability

First we consider dynamics in the $(n = 1)$-dimensional phase space:

$$\frac{dx}{dt} = f(x). \tag{2.6}$$

Small deviations δx from the nonlinear time evolution (2.6) follow the related differential equation

$$\frac{d}{dt}(\delta x) = \left.\frac{df}{dx}\right|_{x_0}\delta x \tag{2.7}$$

which is linearized about the state $x_0 = x(t_0)$ at time t_0. These deviations grow or shrink exponentially with time $t = t_0 + m\tau$, increasing by m time steps of duration τ:

$$\delta x = \delta x_0 \exp(Lm\tau). \tag{2.8}$$

The characteristic exponent (or eigenvalue) L is defined by the Jacobian

$$L = df/dx|x_0 \quad \text{or} \quad d\ln\delta x/dt. \tag{2.9}$$

If $L < 0$ (or >0) the system (2.7) is stable (unstable), $D = |\delta x|$ remains small for all times (or grows exponentially). Vice versa, if $f(x)$ is unknown, too complex, or observed time series are analyzed, the exponential growth rate L may be derived by another prescription, which allows a generalization of characteristic exponents for nonlinear systems. Eliminating L from (2.8) and taking the limit $m \to \infty$ leads to a mean (time averaged) exponential rate of divergence of initially close trajectories (e.g., see Haken, 1983)

$$L(x_0, \delta x_0) = \lim_{m\to\infty} \frac{1}{m\tau} \ln|\delta x|. \tag{2.10}$$

This concept of the Lyapunov characteristic exponent (for trajectories in a one-dimensional phase space) can be generalized to the mean exponential rate of divergence (of two initially close trajectories) evolving in an n-dimensional phase space:

$$L(\mathbf{x}_0, \delta\mathbf{x}_0) = \lim_{m\to\infty} \frac{1}{m\tau} \ln|D(t)|. \tag{2.11}$$

L takes one of the n values $L_1, L_2, \ldots, L_n$, which, in general, would be the largest one. They may be formally related to the n eigenvectors of the Jacobian matrix $\partial f_i/\partial x_j$ leading to a spectrum of Lyapunov exponents (e.g., see Eckmann and Ruelle, 1985):

$$L_1 > L_2 > \ldots > L_n \tag{2.12}$$

Thus, the characteristic exponents refer to the expansion or contraction of different directions in the phase space. In general, the rate of the exponential growth of an infinitesimal vector $\delta\mathbf{x}(t)$ in the n-dimensional phase space is given by the largest of the Lyapunov characteristic exponents L_1; the rate of growth of an infinitesimal surface element is given by the rate of the two largest characteristic exponents $L_1 + L_2$; a k-volume element grows with $L_1 + L_2 + \ldots + L_k$, (e.g., Lichtenberg and Lieberman, 1983). For example, an n-dimensional phase space volume element evolving after (2.1) is represented by the Jacobian determinant $(\det\partial f_i/\partial x_j)$. Thus, the growth rate of the phase space volume element is the growth rate of the Jacobian determinant and given by the sum of all n eigenvalues $L^{(n)} = \sum L_i$, $i = 1, \cdots, n$ (see Eckmann and Ruelle, 1985). For volume conserving (i.e. conservative) systems $L^{(n)} = 0$; for dissipative systems volume is contracted and $L^{(n)} < 0$. On the other hand, a Brownian process yields $L^{(n)} = \infty$, because volume expands infinitely by stochastic motion.

Each positive exponent describes a direction in which the system realizes stretching or divergence decorrelating nearby states. Therefore, the long-term behavior of an initial condition $\mathbf{x}_0$ (with any uncertainty) cannot be predicted; this characterizes a chaotic system with sensitive dependence on initial conditions. Each positive Lyapunov characteristic exponent $L_i > 0$ contributes to the divergence or expansion of a phase-space volume element surrounding the initial state $\mathbf{x}_0$ (i.e., a weather or climate situation); their sum defines an exponential growth rate h of initially small errors (volume elements or ellipsoids)

$$h(\mathbf{x}_0) = \sum_{L_i > 0} L_i \qquad (2.13)$$

and provides a quantitative measure of (un)-predictability; it describes the expansion of an infinitesimal ellipsoid to which only the diverging components ($L_i > 0$) of the principle axes contribute. The inverse value, $1/h$, denotes a mean time scale up to which predictability may be possible; it characterizes a mean time span for e-folding volume expansion of a dynamical system evolving in phase space.

Summarizing: Traditional predictability experiments in meteorology determine the error growth from the evolution of an assumed true state disturbed by a random error perturbation. This provides an estimate of the largest Lyapunov exponent. The expansion of an initial sphere of infinitesimal errors growing into an ellipsoid (evolving in the phase space of the dynamical system) corresponds to all positive Lyapunov characteristic exponents, which give more complete information of the dynamical system's sensitive dependence on initial conditions. To gain this information on predictability the phase space of the dynamical system (2.1) is reconstructed—as described in the following subsection—by using an observed time series of one its variables (defining one coordinate), and the time series of the same variable but shifted by $(m - 1)$ time lags (providing $m - 1$ further coordinates). In this new phase space, spanned by delay coordinates, some geometrical properties of the system's time evolution may then be estimated such as the dimension of the attractor and the divergence of nearby pieces of trajectories (as a measure of predictability).

b. Predictability on attractors

The time evolution of the dynamical system (2.1) is described by a trajectory in the n-dimensional phase space, spanned by the n different variables x_i, $i = 1, \cdots, n$. The position of the trajectory is defined by the components x_i of the state vector

$$\mathbf{x}(t) = (x_1(t), x_2(t), \cdots, x_n(t)). \qquad (2.14)$$

Portraits of the time evolution may show distinct patterns, called attractors that trap trajectories after transients originating from the initial states decreased. These attracting sets often reveal dimensions $d_\infty < n$. Furthermore, attractors do not necessarily have integer dimension; fractal or noninteger dimensions seem to be common for many turbulent systems, which are deterministic with a strong sensitivity on initial conditions and irregular (or chaotic) behavior.

If geometrical properties (such as the dimension of attractors or the characteristic exponents of the flow) are to be evaluated from observations, one is generally confined to time series of a single state variable sampled at regular time intervals Δt. Therefore, Packard et al. (1980; see also Takens, 1981) suggest a reconstruction of the phase space picture of the attractor which can be made plausible by the following arguments. The dynamical system (2.1) for n variables (i.e., a set of n first-order differential equations) can be transformed to a single nonlinear differential equation (of nth order) by eliminating all but one of the variables, say $x(t)$, from (2.1):

$$x^{(n)} = f(x, x', \cdots, x^{(n-1)}) \qquad (2.15)$$

The new trajectory, $\mathbf{x}(t) = (x(t), x'(t), \cdots, x^{(n-1)}(t))$, of the transformed system describes the same dynamics. It evolves in a phase space spanned by the coordinate $x(t)$ plus its $(n - 1)$ derivatives $x'(t), \cdots, x^{(n-1)}(t)$. Instead of this continuous variable $x(t)$ and its derivatives, a discrete time series and its shifts by $(n - 1)$ time lags may be considered (Ruelle, 1981): $\mathbf{x}(t) = [x(t), x(t + \tau), \cdots, x(t + (n - 1)\tau)]$. Choosing τ as the macrotime scale (or autocorrelation time) of the time series $x(t)$ should guarantee linear independence of the delay coordinates. This is plausible because for time lags larger than τ the autocorrelation of the time series tends to zero; i.e., the data become uncorrelated or linearly independent, because a vanishing dot-product defines orthogonality.

Now the attractor of a dynamical system is embedded in a new phase space of delay coordinates without changing its topological (or geometric) properties (e.g., see Fraedrich, 1986). Thus, it is sufficient to (i) consider a single state variable $x(t)$ of a dynamical system and (ii) its trajectory evolving in phase space spanned by time shifted coordinates, if such properties are to be derived from empirical data. In the following two geometrical properties of attractors will be estimated from a statistical analysis of the distances between pairs of points on the trajectory (Grassberger and Procaccia, 1983, 1984): the dimension of attractors and the divergence of independent pieces of trajectories evolving on them.

Consider a pair of points in an m-dimensional phase space of time-lagged coordinates which is sufficiently large to embed the attractor:

$$\mathbf{x}_m(t_i) = (x(t_i), \cdots, x(t_i + (m-1)\tau))$$
$$\mathbf{x}_m(t_j) = (x(t_j), \cdots, x(t_j + (m-1)\tau)). \quad (2.15)$$

They are a distance $r_{ij}(m)$ apart (using Euclidean norm) which depends on the phase-space dimension m:

$$r_{ij}(m) = |\mathbf{x}_m(t_i) - \mathbf{x}_m(t_j)| \text{ with } |t_i - t_j| > \tau. \quad (2.16)$$

The number $N_m(l)$ of such pairs, whose distance is smaller than the prescribed threshold $r_{ij} < l$, is formally determined by

$$N_m(l) = \sum_{i,j=1}^{N} \theta(l - r_{ij}(m)) \quad (2.17)$$

where θ is the Heaviside-function with $\theta(a) = 0$ or 1, if $a >$ or < 0; N is the total number of points. The related cumulative distribution $C_m(l)$ is normalized by the total of N^2 pairs of points:

$$C_m(l) = N_m(l)/N^2. \quad (2.18)$$

This distribution function is an ensemble average over all N points $\mathbf{x}(t) = (x(t), x(t+\tau), \cdots)$ which define a trajectory embedded in the m-dimensional phase space of time-shifted coordinates. Thus, $C_m(l)$ describes the mean relative number of points which occur in a m-dimensional volume element or ball of the radius l, which surrounds every individual state or point of the trajectory $\mathbf{x}(t)$ on the attractor. With increasing distance threshold (or size of the ball) the number $C_m(l)$ of pairs grows; furthermore, $C_m(l)$ changes its shape with increasing embedding dimension m. For $N \to \infty$ the distribution function $C_m(l)$ leads to estimates (i) of the dimension of attractors and (ii) of the divergence of trajectories which evolve on them.

1) Dimension of attractors at fixed embedding dimension m

Consider data points homogeneously distributed on a line (on a surface; in a volume); the number of all pairs of points $C(l)$ which are up to a distance l apart, grows linearly (quadratic or cubic) with increasing l; i.e., proportional to l (l^2; l^3). An attractor is described by a trajectory (of $N \to \infty$ data points) $\mathbf{x}(t) = (x(t), x(t+\tau), \cdots, x(t+(m-1)\tau))$ evolving in a sufficiently high dimensional phase space (of dimension m); its dimension d_∞ scales with the cumulative distance distribution function (2.18) of pairs of points on the trajectory (for $l \to 0$):

$$C(l) \sim l^{d_\infty} \quad (2.19)$$

2) Characteristic exponents at fixed distance threshold l

A state $\mathbf{x}(t_0) = (x(t_0), x(t_0+\tau), \cdots, x(t_0+(m-1)\tau))$ in the m-dimensional phase space of time shifted coordinates defines a piece of time trajectory of the length $(m-1)\tau$. States within a m-dimensional ball of size l surrounding the state $\mathbf{x}(t_0)$ define other pieces of trajectories on the attractor, which remain close (i.e., within a distance $< l$) during the $(m-1)$ time steps. Thus $C_m(l)$ defines the mean relative number of all pairs of pieces of trajectories which stay close together within a distance $< l$ during the time evolution $(m-1)\tau$. They start, however, at different t_0. Thus, averaging is to be taken over all points on the attractor.

Increasing the embedding dimension from m to $m+1$ (but keeping l fixed) prolongs the pieces of trajectories by one time step τ from $(m-1)\tau$ to $m\tau$. Thus, the new cumulative distribution C_{m+1} at fixed l describes the (now reduced) mean number of trajectories which still stay within the distance $< l$; i.e. they remain within balls of size l—the others have escaped. Thus, the change from $(m-1)\tau$ to $m\tau$ or from C_m to C_{m+1} provides a measure for the mean escape rate (divergence) of close pieces of trajectories on the attractor or, which is equivalent, for the Lyapunov characteristic exponents. A scaling of the cumulative distance distribution function C_m by the Lyapunov characteristic exponent (and the sum $h = \sum L_i$, $L_i > 0$) can be deduced.

Consider two pieces of diverging trajectories which remain within a distance threshold ($|\delta\mathbf{x}| < l$) during $(m-1)$ time steps τ. Their chance to diverge beyond the fixed threshold value during the next time step τ rises proportionally to $\exp L m\tau$ [see (2.8)]; vice versa, their chance to remain trapped within the distance threshold decreases proportionally to $\exp(-Lm\tau)$. Furthermore, the chances for an initially large ensemble of trajectories to remain trapped within a volume element (or ellipsoid) decrease proportionally to the expansion along the principle axes, i.e., $\exp(-L_i m\tau)$; note that only diverging axes ($L_i > 0$) contribute. Accordingly, the average relative number C_m of pairs of points with distances $r_{ij} < l$ decreases proportionally to

$$\begin{aligned} C_m &\sim \exp(-m\tau L_1); \exp(-m\tau L_2), \cdots \\ &\sim \exp[-m\tau(L_1 + L_2 + \cdots)] \\ &\sim \exp(-m\tau h). \end{aligned} \quad (2.20)$$

Thus, the predictability h can be interpreted as a mean exponential expansion rate or divergence of pieces of trajectories of the length $m\tau$; averaging occurs over all points of the attractor.

3) Combination and method of analysis

Combination of the proportionalities (2.19) and (2.20) leads to the scaling law of the cumulative distance distribution function $C_m(l)$:

$$C_m(l) \sim l^{d_\infty} \exp(-m\tau h). \quad (2.21)$$

Here it should be noted that a rigorous derivation of this scaling law (e.g., see Eckmann and Ruelle, 1985, based on Pesin, 1977) connects the sum of positive Lyapunov characteristic exponents h of the flow with the mean rate of creation of information or reduction of predictability (i.e., the Kolmogorov–Sinai entropy). It can be approximated (Grassberger and Procaccia, 1984) by $C_m(l)$, which is a measure of the order-2 Renyi entropy. In this sense the unpredictability h (in 2.20) or mean rate of information production defines a lower bound for the mean rate of divergence (2.13) of pieces of trajectories on the attractor (and therefore an upper bound for the predictability time scale). Furthermore, considering empirical data, (2.21) should be interpreted as a limit process with the following requirements: The time series $x(t)$ should be sufficiently long ($N \rightarrow \infty$) after decay of the transients to ensure that the trajectory evolves on the attractor and covers it; the phase space should be of sufficiently high dimensionality ($m \rightarrow \infty$) to embed the attractor, and the balls of size l should be sufficiently small ($l \rightarrow 0$). Note that in calculating the pairwise distances $r_{ij}(m)$, (2.16), the constraint needs to be observed that $|t_i - t_j| > \tau$; i.e. only independent samples (pieces) of trajectories enter the cumulative probability distribution $C_m(l)$. Now it is possible to deduce (i) the dimension of attractors d_∞ and (ii) the mean rate of divergence of (pieces of) trajectories on attractors using the scaling law (2.21) of the cumulative distribution function $C_m(l)$.

The *dimension of attractors* can be deduced from the (linear) slope of the distribution in a $\ln[C(l)]$ versus $\ln l$ diagram

$$d_\infty = \ln[C(l)]/\ln(l) \qquad (2.22)$$

if the dimension m is chosen sufficiently high ($m > m_\infty$) that the attractor is embedded in the phase space of m_∞ time-shifted coordinates. In this sense one obtains the attractor dimension as a saturation value d_∞, which does not change if further coordinates ($m > m_\infty$) are added to the embedding phase space, i.e., if the slopes of the cumulative distributions $C_m(l)$ remain constant for $m > m_\infty$. The dimension d_∞ of the attractor (or, if fractal, its nearest larger integer) provides a measure of the minimum number of independent variables necessary to adequately model the dynamics of the system. The embedding dimension m_∞ gives a reasonable upper bound for the number of variables sufficient to do so. A noninteger attractor dimensionality characterizes chaotic behavior of the time evolution of the dynamical system (with its sensitive dependence on the initial conditions). Furthermore, it provides evidence for the possible existence of a strange attractor, which governs the time evolution and the inherent limit in predictability. However, this estimate of dimensionality does not provide a quantitative measure of the degree of chaos. Therefore, it seems to be relevant to meteorology to also obtain a quantitative estimate of the rate at which predictive ability is lost; this is provided by the sum h of the positive Lyapunov exponents.

The *predictability h on attractors* can be estimated from slopes of the distribution function in a $\ln C_m(l)$ versus $\ln l$ diagram (Fig. 1), if the dimension m is chosen sufficiently high ($m > m_\infty$) that the attractor is embedded in the phase space of time shifted coordinates. The difference between $\ln[C_{m+k}(l)]$ and $\ln[C_m(l)]$ at a fixed-distance threshold l leads to the mean predictability h

$$h \sim \frac{1}{\tau k} \ln \frac{C_m}{C_{m+k}}. \qquad (2.23)$$

The fixed distance l should be selected from an $\ln l$-interval where the related distributions $\ln[C_m(l)]$ can be approximated by straight lines of identical slopes, i.e. where $C(l) \sim l^{d_\infty}$ is satisfied. The inverse value $1/h$ of the mean divergence defines a mean time scale up to which predictability may be possible, if e-folding volume expansion is considered.

3. Application to weather and climate variables

The method of estimating the predictability on attractors is applied to time series of a variable of the weather and the climate system. These are variables, which have been used to estimate the dimensionality of weather and climate attractors (Fraedrich, 1986): Daily values of surface pressure as a single station weather variable (Berlin-Dahlem) and an oxygen isotope record of an Atlantic deep-sea core representing a climate variable. The following results should be interpreted with care, because an insufficient number of data points may lead to systematic errors.

a. *Weather variables*

Only ensembles of winter seasons (of 120 days commencing on 1 November) are analyzed; seasonal time series allow estimates (of a lower bound) of the saturation dimension d_∞, because the related weather attractors can be embedded into phase spaces of the dimension. However, attractors described by trajectories of a 10 to 15 year time period include long-range processes with memory from season to season. Their dimension d_∞ is so large that even embedding dimensions $m_\infty = 15$ to 20 are not sufficient; i.e., the dynamics of these processes is governed by too many degrees of freedom. These estimates are obtained from the slopes $d_\infty = \ln[C_m(l)]/\ln l$ of the cumulative distribution functions $C_m(l)$ plotted in a $\ln[C(l)]$ versus $\ln l$ diagram (see Fig. 1). The slopes $d(m)$ change with increasing embedding dimension m until saturation d_∞ is achieved at m_∞; d_∞ defines the attractor dimension, m_∞ the dimension of the phase space sufficient to embed the attractor. A further increase of the embedding dimension ($m > m_\infty$) does not change the saturation dimension (d_∞). Note that the saturation dimension d_∞ provides a lower bound of the number of

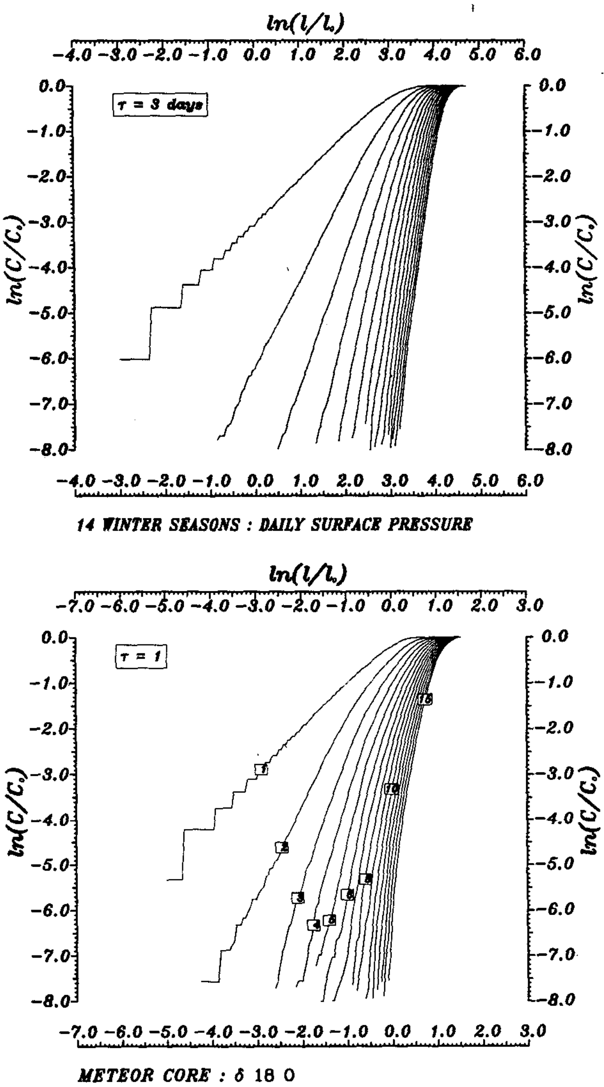

FIG. 1. Cumulative distribution functions $C_m(l)$ for a sequence of embedding dimensions ($m = 1, \cdots, 15$ increasing from left to right). A weather variable (top): daily surface pressure values (at Berlin) for winter seasons. A climate variable (bottom): $\delta\,^{18}O$ data of the Meteor core 13519 covering 775 000 years bp.

essential variables needed to model the dynamics; the upper bound of variables sufficient to do so is estimated by the embedding phase space dimension at saturation, m_∞. For the Berlin surface pressure we obtain $d_\infty \gtrsim 6.8$ to 7.1 and $m_\infty \gtrsim 15$.

Beyond saturation (d_∞, $m > m_\infty$), it is possible to estimate the mean predictability h using (2.23), where m is interpreted as $m > m_\infty$. At fixed-distance threshold l the difference between $\ln C_{m+k}$ and $\ln C_m$ can be obtained from graphs such as shown for the daily surface pressure (Fig. 1); the macro-time scale $\tau = 3$ days is chosen for the weather variable. The predictability estimate leads to an e-folding predictability time scale ($1/h$) of about two weeks (12–15 days). This time scale, (although to be considered as a very weak limit) seems to be in some agreement with the error doubling time of about 8 days deduced from analogues in past observations (Lorenz, 1969; Gutzler and Shukla, 1984); note the conversion factor $\ln 2 = 0.69 \cdots$ from e-folding phase space volume expansion to doubling. Models of the general circulation or hydrodynamic flows (Smagorinsky, 1969; Shukla, 1981; see also Shukla, 1985, for a review) reveal estimates between 2 and 4 days for the doubling time of small errors (the common measure for predictability experiments). This timescale, however, is about 3 to 5 days shorter than our predictability estimates form observed time series on attractors (realizing the conversion $\ln 2 = 0.69$). This discrepancy may be related to the presence of a moderately strong spectral gap in the mesoscale range; such a gap in the energy spectrum would increase the model predictability by about 3 days (Lorenz, 1984b).

b. Climate variables

The analysis of a climate attractor is based on an oxygen isotope record of planctonic species (Sarnthein et al., 1984) gained from a 10.7 m long deep-sea core (*Meteor* 13519) from the eastern equatorial Atlantic. The 182 $\delta\,^{18}O$ values cover 775 000 years BP. They are deduced from 3 to 7 cm slices corresponding to 2000–4000 years of sedimentation. For the embedding of the time (depth) series we use multiples of $\tau = 1$ (or $\tau \sim 3000$ years). The analysis of the cumulative distribution (Fig. 1) leads to a predictability time scale ($1/h$) of 10 to 15 thousand years, provided one accepts the slope $d_\infty \gtrsim 4.4$–4.8 as (the lower bound of) the dimensionality of the climate attractor (See Table 1). Comparable results can be obtained from an analysis by Nicolis and Nicolis (1984), who derived a dimensionality of $d_\infty \sim 3.1$. From their distribution function one derives a predictability time scale ($1/h$) of 17 to 20 thousand years. These estimates should be interpreted with care, because both climate time series consist only of a limited amount of data.

TABLE 1. Estimates of attractor dimensions (d_∞) and time scales of predictability ($1/h$). The predictability time scale is a measure of the mean e-folding phase space volume expansion; for volume doubling multiply with $\ln 2 \sim 0.69$. The estimates are deduced from the cumulative distributions (Fig. 1) applying the scaling law $C_m(l) \sim l^{d_\infty} \exp(-m\tau h)$.

	Estimates	
Variables	Dimension of attractors (d_∞)	Predictability time scale ($1/h$)
Weather variable		
Pressure	≥6.8–7.1	12–17 days
Climate variable		
$\delta\,^{18}O$ *Meteor* core	≥4.4–4.8	10–15 000 years

4. Conclusion

The traditional predictability experiments in meteorology usually determine the error growth of an assumed true state disturbed by a random perturbation. This provides an estimate of only the largest (positive) Lyapunov exponent. All positive Lyapunov exponents correspond to the expansion of an initial sphere of infinitesimal errors growing into an ellipsoid which evolves in the phase space of the dynamical system. In this sense all positive exponents provide a more complete quantitative measure of the degree of chaos (or of the sensitive dependence on initial conditions) and, which is equivalent, of the unpredictability.

Predictability of the weather and climate system is estimated from the evolution of single-variable time series on attractors. These attractors can be embedded in a phase space, which is spanned by coordinates defined by the variable and its successive time shifts. Thus, a point in the phase space represents a piece of a time trajectory (evolving on the attractor), whose duration or length is defined by the dimension of the embedding phase space (i.e., the number of time shifts or phase space coordinates). Thus, points in a ball of size l are the realization of an ensemble of pieces of close trajectories of the same length. Increasing the phase-space dimension (i.e., adding a coordinate with a further time shift of the variable) extends the pieces of trajectories in time. Now, predictability is defined by the divergence of initially close trajectories; this is equivalent to the escape rate of points from a ball of size l, if the time-lengths of the trajectories is stepwise extended, i.e., if the same ball is embedded into a higher-dimensional phase space. Thus, the number difference of pairs of points (in balls of the same size but of dimensionality increasing from m to $m + k$) is equivalent to the number of those initially close pairs of pieces of trajectories which leave the prescribed distance threshold when time progresses (form m to $m + k$ by k time steps). Now the mean predictability can be deduced from this number difference related to the time steps, by which the dimensionality of the embedding phase space is increased (as time progresses); the averaging occurs over balls of the same size l, which surround every point on the attractor. One statistical procedure (suggested by Grassberger and Procaccia, 1984) is based on cumulative number distribution functions of increasing distance l between all pairs of datapoints as they are observed on the attractor; the attractor should be embedded in phase spaces of sufficiently high dimensionality.

The weather variable analyzed seems to evolve on a low dimensional attractor after interannual variability and seasonal variations have been excluded by using only seasonal datasets. The observed predictability time scale can be related to common weather predictability studies which are based on observed analogues and model experiments of error growth. However, limitations of using single-point variables should be noted. They provide only one projection of the weather attractor whereas other further distant points are expected to characterize different regions (on the attractor) and, therefore, may lead to different estimates of predictability time scales. On the other hand, this method may become a useful tool for the analysis of weather and climate models. In particular, regional measures of predictability can be evaluated. The new aspect introduced would be an analysis of model performance in phase space. Finally, it should be noted that both the dimensionality of attractors and the predictability on them can be estimated by the same procedure.

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