

## A Parsimonious Stochastic Water Reservoir: Schreiber's 1904 Equation

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(Manuscript received 26 May 2009, in final form 17 September 2009)

### ABSTRACT

A parsimonious model is presented, leading to Schreiber's aridity–runoff relation as equilibrium solution of the rainfall–runoff chain. The chain commences with a fast stochastic water reservoir of small capacity, representing interception and wetted ground in short time intervals. It feeds a slow (almost stationary) soil moisture reservoir of large capacity, balancing its runoff after long-term averaging. Parameterizing the fast reservoir's capacity by the water equivalent of net radiation available for evaporation leads to a biased coin-flip surrogate for its “full” or “empty” states when rainfall is larger or smaller than the capacity. Rainfall surplus from the fast reservoir's full state feeds the slow (almost stationary) soil moisture reservoir; with the residual evaporating the fast reservoir starts anew as empty. Rainfall below capacity evaporates completely and, leaving the energy surplus for sensible heat, the fast reservoir also starts anew as empty. Employing coin-flip occurrence probabilities from exponentially distributed precipitation yields Schreiber's formula.

### 1. Introduction

More than a century ago, Schreiber (1904) analyzed data of the annual mean discharge  $R_o$  versus the annual precipitation totals  $P$  of continental European river basins fitted to a polynomial curve. Looking at this curve lead him to assume that it can be presented by the formula

$$R_o = P \exp\left(\frac{-N}{P}\right). \quad (1)$$

Schreiber noted on “the physical meaning of the parameter  $N$  [that it] approaches the difference precipitation  $P$  minus runoff  $R_o$  (=evaporation  $E$ ) better for larger precipitation,” which suggests  $N$  to be the water equivalent of the available energy, that is, potential evaporation or net radiation. This formula describes, for a given region, how runoff and evaporation are governed by  $P$  and  $N$  as the two forcing terms of supply and demand. Two regimes emerge because of the limitations of the forcing: energy limitation occurs, if available energy  $N$  is low, so that runoff exceeds evaporation for given precipitation,  $E \sim N$ ; and water limitation occurs, if available energy is so high that water supplied by

precipitation evaporates, which then exceeds runoff,  $E \sim P$ . The ratio  $D = N/P$  defines the aridity index (Budyko 1974), which separates the energy- and water-limited regimes at  $D = 1$ ; it is also used as an indicator for the vegetation types related to climate, such as tundra  $D < 1/3$ , forest  $1/3 < D < 1$ , steppe and savanna  $1 < D < 2.5$ , semidesert  $2.5 < D < 3.5$ , and desert  $3.5 < D$ .

There are more water–energy balance closures like Schreiber's, for example, Ol'dekop (1911; see also Pike 1964; Turc 1954). They are applied in Budyko's (1974) climate analysis centered on the aridity parameter and have been extended to include vegetation explicitly (e.g., Zhang et al. 2004). They are the basis of climate change-related sensitivity studies performed by Dooge (1992; see also Lettau 1969) and paleoclimatic diagnostics (e.g., Kutzbach 1980). Furthermore, they are used to evaluate high-resolution global climate models (Koster and Suarez 1999; Arora 2002) and are directly compared with more complex soil–vegetation–atmosphere–transfer models that, in different climatic zones, find Schreiber's formula very close to observations and model results (Wang and Takahashi 1999). These successful applications and validations of Schreiber's aridity–runoff relations demonstrate their ongoing relevance in climate analysis, in particular for water cycle modeling, sensitivity, and feedback analyses on the regional scale.

Conceptual models of basins responding to water supply by precipitation suggest a probabilistic approach

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(Rodriguez-Iturbe et al. 1990; Hosking and Clarke 1990). From the first concepts (Eagleson 1978; Hosking and Clarke 1990) to a minimalist approach (Milly 1993), all these analyses require a parameter set that outnumbers the three variables entering Schreiber's Eq. (1) and focus on the soil water storage. The following note presents a parsimonious model leading directly to Schreiber's (1904) aridity–runoff relation as equilibrium solution of the rainfall–runoff chain. This chain commences with a fast stochastic water reservoir of small capacity (like interception and wetted ground), which feeds a slow (almost stationary) soil moisture reservoir of large capacity balancing its runoff at the end.

## 2. Stochastic water storage model

The basin model consists of two reservoirs of different residence times and capacities: a fast (and shallow) and a slow (and deep) water reservoir. In the idealized conceptual limit, the fast reservoir acts instantaneously with small capacity, whereas the slow or soil water reservoir has very large capacity and response time:

- (i) The fast water reservoir represents “the first processes in the chain of interlinked rainfall-runoff processes” (Savenije 2004) and plays the role of water in transit before reaching the soil. It is characterized by small capacity, which corresponds to the water equivalent of net radiation available for evaporation in a short (day to week) time interval and an analogously short water residence time. The capacity comprises interception in canopies, herbaceous vegetation, and near the ground in mulch, in wetted ground, and soil layers drying out within the short residence time; even sand without vegetation can intercept water. That is, rainfall exceeding net radiation is a water surplus, which is considered as the fast reservoir's overflow. Therefore, the reservoir is defined as “full” if the rainfall input, which cannot be evaporated during the short time interval, leaves as overflow. Otherwise, it is “empty” with the smaller rainfall being evaporated by the water demanding net radiation. The surplus produced during the full state enters the slow soil moisture reservoir in the same time interval.
- (ii) The slow soil water reservoir represents the last processes in the rainfall–runoff link and, as the moisture capacity is much larger than any fluxes of water during the short time interval, it is considered stationary compared to the effect of these fluxes. Thus, in the climate average, the fast reservoir's surplus balances the slow soil water reservoir's discharge.

- (iii) Two closures connect the fluxes of both reservoirs comprising the catchment: the meteorological closure links atmospheric demand with the fast reservoir's maximum capacity to provide overflow once random water supply by precipitation exceeds net radiative demand (or draft); and the hydrological closure parameterizes the slow soil moisture reservoir's supply by the fast reservoir's overflow (or surplus), which, in the long term mean, compensates the catchment's discharge.

### a. Fast reservoir

The fast reservoir is represented by a parsimonious stochastic storage model evolving at short time intervals that is, for example, on a day-to-day basis. Storage change is forced by random water input from daily to weekly precipitation  $p_k \geq 0$  for  $k = 0, 1, \dots, K$ , which is reduced by a constant water demand (draft)  $N = n_k$  determined by the net radiation as the energy flux available for evaporation. The storage is measured by a water level  $Z(t)$ , which is limited by an upper-bound  $N$  separating an empty ( $0 \leq Z < N$ ) from a full ( $N \leq Z$ ) state. This limit is defined by the water demand (maximum possible evaporation or water equivalent of net radiation), which characterizes the basin's thermal climate. The empty state does not produce a water surplus, whereas the full state generates surplus and, therefore, overflow. The occurrence of a full (empty) reservoir corresponds to intervals with (without) water surplus,  $(p_k - N) > 0$  ( $< 0$ ). The associated probabilities are

$$q_0 = \text{prob}(0 \leq Z < N), \quad q_1 = \text{prob}(N \leq Z). \quad (2)$$

The surplus at full state is assumed to leave the fast reservoir instantaneously to provide the water supply for the slow soil moisture reservoir while the remaining amount evaporates, using up the energy demand. Thus, after the time interval, the full reservoir starts anew as empty. Rainfall less than the energy demand is associated with the reservoir to be empty; there is no surplus, because water evaporates with the remaining net energy, providing the sensible heat flux. Thus, after this time interval, the empty reservoir starts also anew as empty. In the long term mean, the surplus (or overflow) balances the slow soil moisture reservoir's discharge (runoff). In this sense the fast water reservoir's full and empty states evolve like a biased coin flip experiment with interval occurrence probabilities (2),  $q_0$  and  $q_1$  with  $q_0 + q_1 = 1$ , which are parameterized as follows.

### b. Meteorological closure

The stochastic forcing of the fast reservoir comprises the short (daily to weekly) interval water supply by

precipitation  $p_k > 0$  for  $k = 0, 1, \dots, K$ , and the related constant energy demand  $N = n_k$ . To a good approximation, the short interval precipitation intensities are exponentially distributed and assumed to occur independently,

$$\text{prob}(p_k \leq p^*) = 1 - \exp\left(\frac{-p^*}{P}\right). \quad (3)$$

Here, the interval mean precipitation  $P$  is deduced from the total water input of  $K$  intervals  $S(K) = \sum_k^K p_k$ , so that the mean is estimated to be  $P = S(K)/K$ . For coupling with the parsimonious water storage scheme, the continuous exponential distribution function is approximated by a discrete binary distribution with two complementary probabilities. They are associated with two mutually exclusive classes of precipitation: smaller-equal or larger than the threshold  $p^* = N$  prescribed by the net radiation:

$$\begin{aligned} \text{prob}(p_k \leq N) &= 1 - \exp\left(\frac{-N}{P}\right), \\ \text{prob}(p_k > N) &= \exp\left(\frac{-N}{P}\right), \end{aligned} \quad (4)$$

with  $\text{prob}(p_k \leq N) + \text{prob}(p_k > N) = 1$ . Note that given the thresholding (4), the effect of zero precipitation does not render relevant, as rainfall is only important if  $P > N$ , and so very small rainfall amounts can effectively be considered zero.

Now the random atmospheric water supply and demand are combined with the occurrences of the reservoir's states, quantifying their probabilities:

$$\begin{aligned} q_0 &= \text{prob}(p_k \leq N) = 1 - \exp\left(\frac{-N}{P}\right), \\ q_1 &= \text{prob}(p_k > N) = \exp\left(\frac{-N}{P}\right). \end{aligned} \quad (5)$$

In summary, the biased coin flip is introduced as a surrogate of the fast water reservoir to simulate the occurrence probabilities of binary water storage states empty or full, that is, with or without water surplus. The capacity  $N$ , which separates the two states, is prescribed by the reservoir's climatological embedding in terms of the exponentially distributed precipitation reduced by net radiation. This quantifies the coin flip's bias in terms of the aridity index  $D = N/P$ , which characterizes water- and energy-limited conditions (at  $D = 1$ ) and, subsequently, the climate and vegetation properties of the basin. In this sense the available energy  $N$  provides a natural upper bound of the fast reservoir's storage.

### 3. Schreiber's equation

The climate mean water budget components of the basin can now be expressed in terms of the occurrence

probabilities of the full or empty intervals determined by the biased coin-flip trial, which represents the evolution of the fast reservoir. The discharge of the stationary or slow reservoir is linked to the occurrence probability of the fast reservoir's full state and its corresponding water surplus which, entering the slow reservoir at the same time interval, balances its mean runoff Ro:

$$\text{Ro} = Pq_1 = P \exp\left(\frac{-N}{P}\right). \quad (6)$$

This result is easily demonstrated: The climate mean runoff Ro is determined by the occurrence probability of the full state, which, in the climate mean  $\langle \cdot \rangle$ , yields

$$\begin{aligned} \text{Ro} &= \langle p_k \rangle_{\text{full}} - \langle N_k \rangle_{\text{full}} \\ &= \int_N^\infty \frac{p_k \exp(-p_k/P) dp_k}{P} - \int_N^\infty \frac{N \exp(-p_k/P) dp_k}{P} \\ &= (P + N) \exp\left(\frac{-N}{P}\right) - N \exp\left(\frac{-N}{P}\right) \\ &= P \exp\left(\frac{-N}{P}\right) = Pq_1. \end{aligned} \quad (7)$$

The basin's equilibrium water balance  $\text{Ro} = P - E$  can be analogously formulated in terms of the fast reservoir's equilibrium state probability of emptiness:

$$E = Pq_0 = P(1 - q_1). \quad (8)$$

Thus, introducing the evaporation ratio  $E/P$  as a function of  $D = N/P$  yields the more common way to write Schreiber's Eq. (1) using the basin's aridity index and thus the catchments climatological setting:

$$\frac{E}{P} = 1 - \exp(-D) = q_0. \quad (9)$$

In this sense Schreiber's formula can also be interpreted in probabilistic terms as an equation of state, describing the evaporation ratio as an occurrence probability of (short) time intervals of "emptiness," which depends on the aridity index  $D$ . For example, at the border between energy- and water-limited climate conditions,  $D = 1$ , the fast reservoir attains an empty state interval on 63% of the biased coin flips.

Furthermore, the biased coin flip as a surrogate for the fast reservoir can also be interpreted as a two-state Markov chain evolving not unlike the truncated multistate Moran dam (Moran 1959; see also Gani 1955; Langbein 1961; Lloyd 1963; Prabhu 1980). If the transitions (i) are prescribed by coin-flip state probabilities (2), and (ii) impose an empty (full) reservoir state when a decrease (increase)

in water content is impossible, then an equilibrium two-state Markov chain is obtained, predicting future in terms of the mean occurrence probabilities.

#### 4. Concluding remarks and outlook

A parsimonious stochastic water storage model of the rainfall–runoff chain leads to the first empirically deduced aridity–runoff relations as an equilibrium or climate mean solution. Combining only three climate variables—runoff, precipitation, and net radiation—it is known as the evaporation ratio curve or Schreiber's (1904) formula and is applicable in many climates of the earth.

For the stochastic model to be parsimonious, the following four simplifications are made. (i) A two-time scale approach separates a fast stochastic water reservoir of small capacity from the slow (stationary) soil moisture reservoir balancing the runoff. (ii) A biased coin flip represents the fast reservoir generating water surplus (or overflow) on short time intervals. (iii) A meteorological closure links the likelihood of water surplus to the fast reservoir's small capacity, which is parameterized in terms of the water equivalent of energy available for evaporating the exponentially distributed precipitation input, which is accumulated during a short time interval. (iv) The slow reservoir has a large capacity, so that it can be assumed stationary. Thus, the input of the fast reservoir's surplus is balanced by the mean runoff.

Schreiber's formula is interpreted in terms of a fast biased coin flip (or Markov-equilibrium state) approach for various kinds of vegetation covering slow soil water reservoirs. The slow reservoir's response to the random input is considered as an averaging process for the rainfall–runoff chain providing the climate mean runoff. Deviations from the climate mean are not considered here, although low-frequency variability, long-term memory, and scaling in river runoff or moisture storage are important ingredients of the rainfall–runoff chain. The Ansatz presented here may be extended to generate long-term memory.

*Acknowledgments.* The referee's comments are appreciated. Thanks are due to Kevin P. Tu (Dawson Lab, Berkeley, CA) for providing information on the "fast reservoir." Support has been provided by the Max Planck Fellowship.

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