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Topographic Rossby waves over Antarctica

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ABSTRACT

The linear barotropic vorticity equation on an infinite polar f-plane is solved for free eigenmodes supported by the zonally averaged topography of Antarctica. Analytic solutions are derived for an exponential orographic profile. Numerical methods are used to obtain results for a realistic profile. The structure and frequency of these topographic Rossby waves are discussed and compared to observations.

1. Introduction

Rossby waves are found in fluids with gradients of the potential vorticity. For topographic Rossby waves, the necessary gradient of potential vorticity is provided by the variation of the local depth of the fluid. In lakes, for example, these waves are supported by the sloping bottom topography (e.g., Saylor et al., 1980; Stocker and Hutter, 1985). Topographic Rossby waves in the atmosphere are tied to the slopes of the major mountain massifs. In particular, one would expect that the Antarctic ice dome with its enormous extent and its gentle slopes provides an almost ideal setting for topographic Rossby waves. In this note, we consider the structure and dispersion relation of barotrophic topographic Rossby waves above the Antarctic ice dome.

2. Equations

We consider the motion of a barotropic fluid on an infinite polar f-plane with coordinates r (distance to the pole), longitude λ and height z. The depth of the fluid is determined by the topography h_b of the ice dome, assumed to be axisymmetric, and a rigid lid on top at height $z = h_t$ which models the tropopause. As depicted in Fig. 1 the orographic profile is restricted to the domain $r \leqslant r_0$ so that $H = h_t - h_b =$ constant, for $r > r_0$.

To satisfy the equation of continuity

$$0 = \frac{H}{r}\left(\frac{\partial}{\partial r}(rv) + \frac{\partial}{\partial \lambda}u\right) + v\frac{\mathrm{d}H}{\mathrm{d}r}, \tag{2.1}$$

with $v = \mathrm{d}r/\mathrm{d}t$, $u = r\mathrm{d}\lambda/\mathrm{d}t$, we introduce the transport stream function ψ:

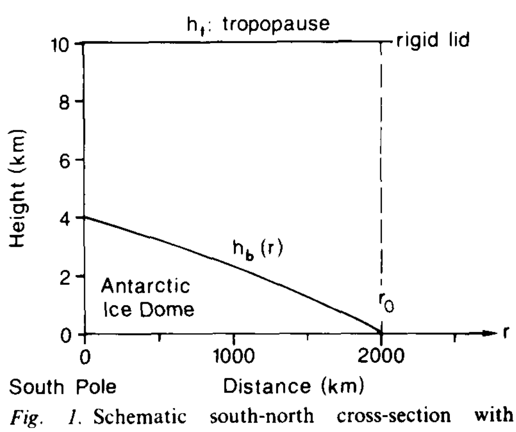

Fig. 1. Schematic south-north cross-section with bottom topography.

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$$Hv = \frac{1}{r}\frac{\partial\psi}{\partial\lambda}, \qquad Hu = -\frac{\partial\psi}{\partial r}. \tag{2.2}$$

The potential vorticity equation when linearized about a state of solid rotation $\Omega_0 = \bar{U}/r$, takes the form

$$\left(\frac{\partial}{\partial t} + \frac{\bar{U}}{r}\frac{\partial}{\partial\lambda}\right)\left\{\frac{1}{r}\frac{\partial}{\partial r}\frac{r}{H}\frac{\partial\psi'}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\lambda}\left(\frac{1}{H}\frac{\partial\psi'}{\partial\lambda}\right)\right\}$$

$$= \frac{1}{rH}\frac{\partial\psi'}{\partial\lambda}\left(\frac{f}{H}\frac{\mathrm{d}H}{\mathrm{d}r}\right), \tag{2.3}$$

where $f < 0$ is the Coriolis parameter and ψ' is the perturbation stream function.

To solve (2.3), we have to impose proper boundary conditions. We consider two cases. First we study the topographic Rossby waves which are completely confined to the Antarctic domain. Then we have to require $\psi' = 0$ at $r = r_0$. This case corresponds with the problem of a closed circular basin treated by Lamb (1895) and Saylor et al. (1980). Holton (1971) used a rotating tank to study forced Rossby waves in such a domain. Next we proceed to consider Rossby waves on an infinite f-plane with $\psi' \to 0$ for $r \to \infty$.

3. Solutions

3.1. Antarctic domain

We introduce

$$\psi' = r^k \phi(r)\exp(ik\lambda - i\omega t) \tag{3.1}$$

in (2.3) to arrive at

$$r\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{\mathrm{d}\phi}{\mathrm{d}r}(2k + 1 - \varepsilon r)$$

$$- \phi\varepsilon\left(k - \frac{fk}{\omega - k\Omega_0}\right) = 0, \tag{3.2}$$

where $\varepsilon = \mathrm{d}\ln H/\mathrm{d}r$. We choose an orographic profile with $\varepsilon = \mathrm{const.}$:

$$h_b = h_t - (h_t - h_b)_{r=0}\exp(\varepsilon r) \tag{3.3}$$

for $r \leqslant r_0$. Then (3.2) is Kummer's equation for $r \leqslant r_0$ (e.g., Kamke (1951)) which is solved by the confluent hypergeometric function

$$\phi(r) = \phi_0 M(a, b, \varepsilon r) \tag{3.4}$$

provided we can choose the parameter $a = k(1 - f/(\omega - k\bar{U}/r))$, $b = 2k + 1$ such that $M(a, b, \varepsilon r_0) = 0$ (see also Lamb, 1895) is satisfied.

Using the approximate formula

$$\varepsilon r_0 \sim \pi^2(m + \tfrac{1}{2}b - \tfrac{3}{4})^2/(2b - 4a) \tag{3.5}$$

for the position εr_0 of the mth positive zero of M (e.g., Abramowitz and Stegun, 1964) we find the dispersion relation

$$\omega \sim \frac{k\bar{U}}{r} + \frac{4fk}{(\pi^2/\varepsilon r_0)(m - \frac{1}{4} + k)^2 - 2}. \tag{3.6}$$

With $h_b = 4000$ m at $r = 0$, $h_b = 0$ at $r = 2000$ km, and $h_t = 10{,}000$ m, we obtain $\varepsilon r_0 = 0.5$. Then, the denominator in (3.6) is positive for all k, $m > 0$ and, therefore, all topographic Rossby waves are retrograde for $\bar{U} = 0$. The period $T = |2\pi/\omega|$ increases with m, k and decreases with increasing "steepness" εr_0 of the topographic profile. Not surprisingly, then, we arrive at a result which bears strong resemblance to the dispersion relation for conventional β-plane Rossby waves. We want to point out, however, that the topographic β-parameter $|f\varepsilon|$ is at least one order of magnitude larger than $\beta = \mathrm{d}f/\mathrm{d}r$ in the Antarctic domain. One would expect, therefore, that the topographic effect is more important than the curvature of the earth.

A zonal wavenumber frequency diagram (Fig. 2a) can be constructed from (3.6) to display the dispersion line $\omega(k, m)$. The zonal wind $\bar{U}$ should be regarded as a vertical and zonal average. Observations suggest small values for $\bar{U}$ because easterlies dominate in the lower, and westerlies prevail in the upper troposphere (see e.g. Oort, 1983). The zonal wavenumber-frequency diagram (Fig. 2a) shows dispersion lines $\omega(k)$ for eastward and westward ($\omega >$ and < 0) phase propagations of topographic Rossby waves for various zonal winds $\bar{U}$ and meridional (or radial) wavenumbers (m). The related phase velocities c also refer to the latitude circle at $r_0 = 2000$ km distance from the pole.

For vanishing zonal wind $\bar{U} = 0$, the largest mode with $m = 1$ and $k = 1$ has a period of 7.1 days; small modes with k, $m > 3$ are extremely slow and have periods of a month and longer. Note that the group velocity $c_g = \partial\omega/\partial k$ changes sign for $k \sim |m - \frac{1}{4}|$. All modes are retrograde. For $\bar{U} < 0$, the westward phase speed is increased, of course. For the weak westerlies ($\bar{U} \sim 5$ m/s), as they are observed for the vertically integrated zonal wind speeds over Antarctica (Oort and Peixoto, 1983), only smaller wavenumbers (larger modes) remain retrogress-

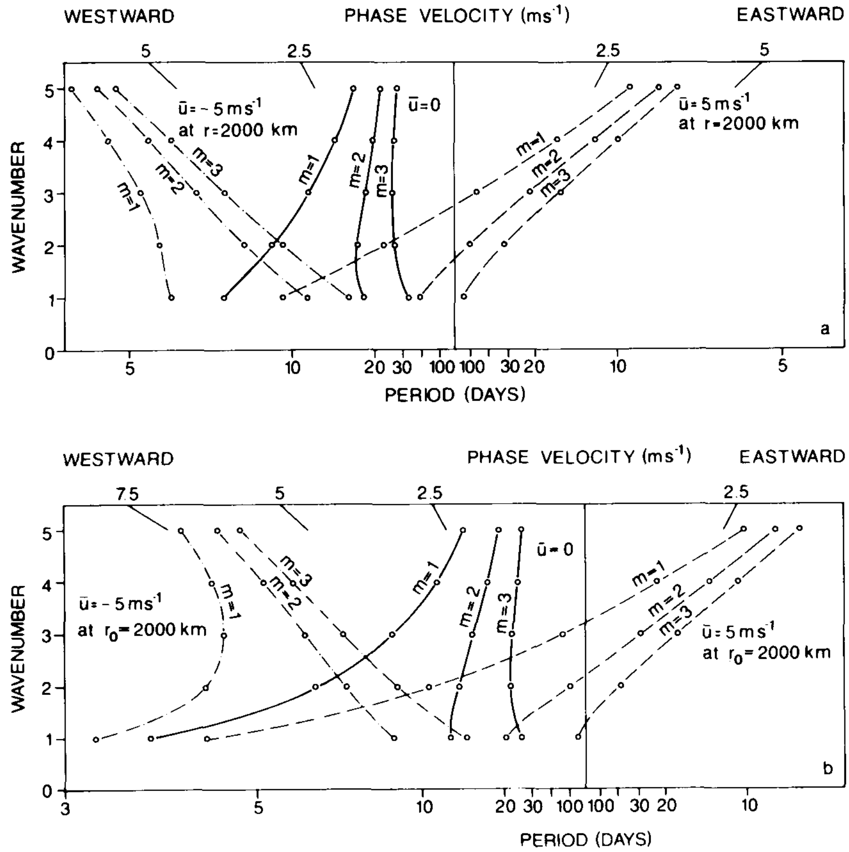

Fig. 2. Dispersion lines in a zonal wavenumber-frequency diagram of topographic Rossby waves for different rates of solid rotation $\Omega_0 = \overline{U}/r$; $\bar{u} = \Omega_0 r_0$ refers to a vertical mean zonal velocity at $r_0 = 2000$ km distance from the pole. The linear frequency axis is labelled in period of days $T = |2\pi/\omega|$. The upper abscissa gives the related phase velocity $c = L/T$ which refers to wavelengths $L = 2\pi r_0/k$ on a circle of $r = 2000$ km distance from the pole, i.e., at the edge of the ice dome. At $r = 1000$ km distance, the phase velocities c and mean zonal winds $\overline{U}$ would attain half of the values given in this figure. Top (a): bounded domain; bottom (b): infinite f-plane.

ive, the larger, ones, however, are progressive with modest phase speeds.

3.2. Infinite f-plane

Outside of Antarctica, we obtain from (2.3)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi'}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi'}{\partial\lambda^2} = 0, \tag{3.7}$$

i.e., we have irrotational flow. The harmonic wave solution to (3.7) is

$$\psi' = \psi_0 r^{-k}\exp(ik\lambda - i\omega t), \tag{3.8}$$

where the boundary condition $\psi' \to 0$ for $r \to \infty$ has been taken into account. We have to match solutions of (3.2) and (3.7) at $r = r_0$, requiring continuity of ψ' and $\partial\psi'/\partial r$ at $r = r_0$. It is no problem to work out the solution in terms of hypergeometric functions. The resulting expression

$$\frac{r}{M}\frac{\mathrm{d}M}{\mathrm{d}r}\bigg|_{r=r_0} = -2k \tag{3.9}$$

with the confluent hypergeometric function $M = M(a, b, \varepsilon r)$ requires after some algebra (Abramowitz and Stegun, 1964, eqs. 13.4.3 and 13.4.10) that $M(a, b-1, \varepsilon r_0) = 0$. Now the approximation for the position εr_0 of the mth zero of $M(a, b-1, \varepsilon r_0)$ yields an explicit dispersion relation

$$\omega \sim \frac{k\bar{U}}{r} + \frac{4kf}{(\pi^2/\varepsilon r_0)(m+k-\frac{3}{4})^2}, \qquad (3.10)$$

which is displayed in Fig. 2b. The largest mode with $m = k = 1$ has a period of 3.8 days for $\bar{U} = 0$ which is about half of the period deduced for the bounded domain. Again, smaller modes with k, $m > 3$ are extremely slow and have periods of about a month or more. It is easy to show that the periods $2\pi\omega^{-1}(k,m)$ on the infinite f-plane are always shorter than the corresponding periods in the closed Antarctic domain. This was to be expected since the speed of Rossby waves increases with their scale. For $\bar{U} = 0$, the group velocity vanishes if $k = |m - \frac{3}{2}|$.

The profile (3.3) has been chosen because of its mathematical convenience. As can be seen from Fig. 1, (3.3) gives an almost linear increase of fluid depth with distance to the pole. To solve (2.3) for more realistic orography we have to use numerical methods. We discretize (2.3) in the domain $0 \leqslant r \leqslant r_0$:

$$\left[\frac{r_{j+\frac{1}{2}}(\hat{\psi}_{j+1} - \hat{\psi}_j)}{H_{j+\frac{1}{2}}} - \frac{(\hat{\psi}_j - \hat{\psi}_{j-1})r_{j-\frac{1}{2}}}{H_{j-\frac{1}{2}}}\right]\frac{1}{r_j \mathrm{D}r}$$

$$-\frac{k^2\hat{\psi}_j}{r_j^2 H_j} - \frac{k\hat{\psi}_j(H_{j+\frac{1}{2}} - H_{j-\frac{1}{2}})f\lambda}{\mathrm{D}r \cdot r_j \cdot H_j^2} = 0, \qquad (3.11)$$

where j is the grid point index, $\mathrm{D}r$ the grid increment and, correspondingly, $r = \mathrm{D}r(j-1)$. In writing down (3.11) we have assumed $\psi' = \hat{\psi}(r)\exp(-i\omega t + \lambda k)$ and, therefore, $\lambda = (\omega - \Omega_0 k)^{-1}$. The boundary condition at $r = r_0$ is posed in analogy to (3.9):

$$\psi_J - \psi_{J-1} = -k\psi_J \mathrm{D}r/r_0, \qquad (3.12)$$

where J is the index of the grid point at $r = r_0$.

Eq. (3.10) with the appropriate boundary conditions poses a self-adjoint Sturm-Liouville problem (e.g., Kamke (1951)) in discrete form. Correspondingly we expect to obtain as many real eigenvalues λ_{km} with $\lambda_{km} < \lambda_{km+1}$ as there are grid points. The related eigenfunctions have $m - 1$ zeros in the domain $0 < r < r_0$. We choose $J = 22$ and present the resulting periods $T_{km} = 2\pi\lambda_{km}$ for selected eigenfunctions $\varphi_{km}(r)$ in Table 1. For intercomparison with (3.10) we first solve (3.11) with (3.3) as the orographic profile. The period of the spatially largest mode is now 3.6 days as compared to $T = 7.1$ days in the case of a closed domain. The period increases with increasing m, of course. In brackets we give the periods according to (3.10). The agreement of the analytical and numerical results deteriorates with increasing k and m. We have to keep in mind, however, that (3.10) is an approximation and that solutions of (3.11) are subject to numerical errors which increase with increasing m. Thus perfect agreement should not be expected.

The eigenfunctions φ_{11}, φ_{12}, φ_{21}, φ_{13} are depicted in Fig. 3. For $r > r_0$, all the eigenfunctions decrease $\sim r^{-k}$ and φ_{12} has one zero in the Antarctic domain. We solved (3.11) for a strictly linear orographic profile. It turned out that the corresponding solutions are quite close to that for the profile (3.3). For example, one obtains $T_{11} = 4d$, $T_{21} = 5d$, $T_{31} = 7d$, $T_{41} = 8d$ in close agreement with Table 1. Matters are different when we consider a more realistic representation of the Antarctic orography. By performing a zonal average over the topography of Antarctica we obtain a profile h_b which comes as close as possible to reality. As compared to (3.3) the slope of the terrain is more gentle near the pole but becomes relatively steep near the coast line. The resulting periods are given in Table 1. It is seen that topographic Rossby waves above the Antarctic terrain move more slowly than those above (3.3). The corresponding eigenfunctions are similar to those presented in

Table 1. *Periods (days) of the eigenfunctions φ_{km} of (3.10) with $\bar{U} = 0$, k = zonal wavenumber, m = mode number; number of gridpoints $J = 22$; first number: orographic profile (3.3) with $h_t = 10{,}000$ m; $h_b = 4000$ m at $r = 0$*

m \ k	1	2	3	4
1	4 (4), 5	5 (6), 7	6 (8), 9	7 (11), 10
2	12 (12), 15	11 (13), 12	12 (14), 13	13 (17), 15
3	24 (25), 47	20 (22), 32	20 (22), 28	20 (23), 27
4	41 (43), 78	30 (33), 52	28 (31), 73	26 (31), 33

The numbers in brackets refer to the approximate analytic dispersion relation (3.10). The second number gives the period for a realistic profile of the Antarctic ice dome with $J = 24$.

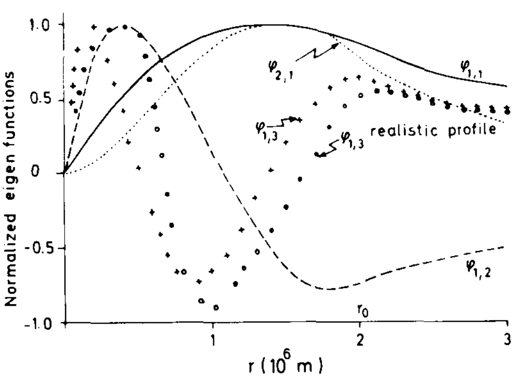

Fig. 3. Discrete eigenfunctions φ_{km} of (3.11) for the topographic profile (3.3); k: zonal wavenumber, m: number of eigenvalue. The eigenfunctions are normalized to have a maximum $\varphi_{km} = 1$. The number of gridpoints is $J = 22$ in the domain $0 < r < r_0$. The circles give the eigenfunctions φ_{13} for a realistic profile of the Antarctic ice dome.

Fig. 3. As an example we give φ_{13} (circles) in Fig. 3. The maximum of φ_{13} is somewhat to the north of that of the eigenfunction for (3.3) and the minimum is deeper. This means that the analytic solution of the problem provides a reasonably good approximation.

4. Discussion

We presented the free topographic modes of the barotropic vorticity equation for a closed domain and also for the case of an infinite f-plane. Approximate dispersion relations have been derived for the profile (3.3) which compared favorably with the corresponding numerical results. The structure of the free modes depends to some extent on the specific assumptions underlying the basic equation. For example, we can replace the rigid lid at $z = h_t$ by a free surface or we can increase the height of the tropopause. Corresponding computations showed, however, that the solutions are fairly robust with respect to such changes. It is more difficult to assess the influence of the curvature of the earth on the results. We have extended (3.2) to include the conventional β-effect. As had to be expected, β has almost no influence on the modes for the case of a closed Antarctic domain. As for the infinite domain we have free Rossby modes which are only partly influenced by the topography of Antarctica. To obtain meaningful results one would have to consider a spherical domain. It has been felt that this problem was beyond the scope of this note.

Is there observational evidence of topographic Rossby waves over Antarctica? Spekat and Fraedrich (1983) present vertical phase and amplitude profiles of short period ($T \leqslant 5$ days) disturbances obtained from Halley Bay (75°S, 26°W) rawinsonde observations, which reveal barotropic structures dominating the dynamics in the troposphere above the shallow boundary layer: the disturbances show small tilts of the vertical axes of geopotential height and meridional wind velocity, and the vertical changes of the related amplitudes are small. Quite recently, Daley and Williamson (1985) provided evidence of distortions of global Rossby modes over Antarctica. We speculate that these distortions are caused by the orography.

To eventually provide further evidence we have to consider the data requirements for a verification of the theoretical results. We found that the Rossby wave dispersion is fairly sensitive to (i) boundary conditions (Fig. 2; (3.6), (3.10)), (ii) orographic profile (Table 1, Fig. 3) and (iii) zonal wind (Fig. 2); in particular, the frequency strongly depends on the meridional (radial) wave number m. Thus it is not sufficient to verify the Rossby modes along a latitude circle but one has to observationally include the meridional structure over the ice dome. Although there is a rather dense rawinsonde network along the Antarctic coast, data coverage in the interior is sparse. Thus it is only possible to deduce the zonal wave number and/or frequency structure at the coast (as attempted by Spekat and Fraedrich, 1983; see also Daley and Williamson (1985)). However, it is not possible to deduce the meridional structure from the available data which is necessary for a verification.

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REFERENCES

Abramowitz, M. and Stegun, I. A. 1964. *Handbook of mathematical functions*. National Bureau of Standards, Washington, D.C., 504–515.

Daley, R. and Williamson, D. L. 1985. The existence of free Rossby waves during January 1979. *J. Atmos. Sci. 42*, 2121–2141.

Holton, J. 1971. An experimental study of forced barotropic Rossby waves. *Geophys. Fluid Dyn. 2*, 323–341.

Kamke, E. 1951. *Differentialgleichungen*, Bd. 1. Akad. Verlagsges., Leipzig, 666 pp.

Lamb, H. 1895. *Hydrodynamics*. Cambridge University Press.

Oort, A. H. and Peixoto, J. P. 1983. Global angular momentum and energy balance requirements from observations. In *Theory of climate* (ed. B. Saltzman). Adv. Geophys. (Academic Press), *25*, p. 355.

Oort, A. H. 1983. Global atmospheric circulation statistics, 1958–1973. NOAA Professional Paper, *14*, US Dept. of Commerce, 180 pp.

Saylor, J. H., Huang, C. K. and Reid, R. O. 1980. Vortex modes in Southern Lake Michigan. *J. Phys. Oceanogr. 10*, 1814–1823.

Spekat, A. and Fraedrich, K. 1983. Further studies on single station climatology: (iii) Time spectral analysis of Halley Bay (Antarctic) rawinsonde data. *Contrib. Atmos. Phys. 56*, 213–220.

Stocker, Th. and Hutter, K. 1985. A model for topographic Rossby waves in channels and lakes. Mitt. Versuchsanstalt für Wasserbau, Hydrologie und Glaziologie, Nr. 76. Eidgenössische Technische Hochschule, Zürich, Switzerland, available upon request.